1. The interaction between two populations with densities N_1 and N_2 is modelled by

$$\begin{aligned}\frac{\mathrm{d}N_1}{\mathrm{d}t} &= rN_1\left(1-\frac{N_1}{K}\right) - aN_1N_2\left(1-\exp[-bN_1]\right), \\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= -dN_2 + eN_2\left(1-\exp[-bN_1]\right),\end{aligned}$$

where a, b, d, e, r and K are positive constants.

   (a) What type of interaction exists between N_1 and N_2? What do the various terms imply ecologically?

   (b) Non-dimensionalise the system by writing

$$u = \frac{N_1}{K}, \qquad v = \frac{aN_2}{r}, \qquad \tau = rt, \qquad \alpha = \frac{e}{r}, \qquad \delta = \frac{d}{r}, \qquad \beta = bK.$$

   (c) Determine the non-negative equilibria and note any parameter restrictions.

   (d) Discuss the linear stability of the equilibria.

   (e) Show that a non-zero N_2 population can exist if $\beta > \beta_c = -\ln(1-\delta/\alpha)$.

   (f) Briefly describe the bifurcation behaviour as β increases with $0 < \delta/\alpha < 1$.

2. Consider a lake with some fish attractive to fishermen. We wish to model the fish-fishermen interactions under the following assumptions:

   - the fish population grows logistically in the absence of fishing;
   - the presence of fishermen depresses the fish growth rate at a rate jointly proportional to the size of the fish and fisherman populations;
   - fishermen are attracted to the lake at a rate directly proportional to the number of fish in the lake;
   - fishermen are discouraged from the lake at a rate directly proportional to the number of fishermen already there.

   (a) Write down a mathematical model for this situation, clearly defining your terms.

   (b) Show that a non-dimensionalised version of the model is

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\tau} &= ru(1-u) - uv, \\ \frac{\mathrm{d}v}{\mathrm{d}\tau} &= \beta u - v,\end{aligned}$$

   where u and v represent the non-dimensionalised fish and fishermen populations, respectively.

   (c) Calculate the steady states of the system and determine their stability.

   (d) Draw the phase plane, including the nullclines and phase trajectories.

   (e) What would be the effect of adding fish to the lake at a constant rate?

3. (a) What kind of interactive behaviour between two populations, N_1 and N_2, is suggested by the model

$$\begin{aligned}\frac{\mathrm{d}N_1}{\mathrm{d}t} &= r_1 N_1\left(1-\frac{N_1}{K_1+b_{12}N_2}\right),\\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= r_2 N_2\left(1-\frac{N_2}{K_2+b_{21}N_1}\right),\end{aligned}$$

where r_1, r_2, K_1, K_2, b_{12} and b_{21} are positive constants?

(b) Show that, with appropriate non-dimensionalisation, this model takes the form

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}\tau} &= u_1\left(1-\frac{u_1}{1+\alpha_{12}u_2}\right),\\ \frac{\mathrm{d}u_2}{\mathrm{d}\tau} &= \rho u_2\left(1-\frac{u_2}{1+\alpha_{21}u_1}\right),\end{aligned}$$

where

$$u_1=\frac{N_1}{K_1}, \qquad u_2=\frac{N_2}{K_2},$$

τ is non-dimensionalised time and α_{12}, α_{21} and ρ are positive parameters.

(c) Determine the steady states and their linear stability, taking care to list any restrictions on parameters.

(d) By drawing the nullclines and sketching phase trajectories, briefly discuss the behaviour of the model for the cases $\alpha_{12}\alpha_{21}<1$ and $\alpha_{12}\alpha_{21}>1$.

4. An animal population is prone to a fatal disease. There is a limited vaccine that creates immunity in the susceptible population but has no effect on infected animals. The higher the number of infected animals observed, the more vigorously vaccinations are administered.

Let I denote the number of infected animals, S the number of susceptible animals, V the number of vaccinated animals and R the number of dead animals. An ordinary differential equation description of these interactions can be written

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta SI - pSI,\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \beta SI - aI,\\ \frac{\mathrm{d}R}{\mathrm{d}t} &= aI,\\ \frac{\mathrm{d}V}{\mathrm{d}t} &= pIS.\end{aligned}$$

(a) Explain the biological interpretation of each of the terms in the model.

(b) Non-dimensionalise the model to give

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\tau} &= -si(1+\eta),\\ \frac{\mathrm{d}i}{\mathrm{d}\tau} &= i(s-1),\\ \frac{\mathrm{d}r}{\mathrm{d}\tau} &= i,\\ \frac{\mathrm{d}v}{\mathrm{d}\tau} &= \eta is,\end{aligned}$$

where the non-dimensionalised populations are indicated using corresponding lower-case letters, τ is non-dimensional time and the parameter $\eta > 0$ should be given.

(c) Suppose that the initial conditions are

$$s(0) = s_0, \qquad i(0) = i_0, \qquad r(0) = 0, \qquad v(0) = 0,$$

where $s_0 > 0$ and $i_0 > 0$, and assume that $i \to 0$ as $t \to \infty$.

i. By considering $\mathrm{d}i/\mathrm{d}s$ and $\mathrm{d}v/\mathrm{d}s$, determine the non-dimensionalised number of animals that catch the disease and thus die. The answer should be in terms of i_0, s_0, η and $s(\infty)$.

ii. For $\eta \gg 1$ show that the number of animals who die from the disease is approximately $i_0 + s_0/\eta$.