Special Relativity

Trinity Term 2017

Problem sheet 2

1. **Lorentz transformations and velocity.** Let O and O' be two non-accelerating observers whose inertial coordinate systems are related by a proper orthochronous Lorentz transformation

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = L \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} .$$

Show that the Lorentz transformation matrix L must be of the form

$$\begin{pmatrix} \gamma & -\gamma v_1'/c & -\gamma v_2'/c & -\gamma v_3'/c \\ \gamma v_1/c & * & * & * \\ \gamma v_2/c & * & * & * \\ \gamma v_3/c & * & * & * \end{pmatrix},$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity of observer O' in frame O, $\mathbf{v}' = (v_1', v_2', v_3')$ is the velocity of observer O frame O', and $\gamma = \gamma(v) = \gamma(v')$.

2. **Lorentz matrices.** Which of the following matrices represent Lorentz transformations? Which are proper? Which are orthochronous?

$$\begin{pmatrix} \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -2 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

3. **Geometry of four-vectors.** Show that:

   (i) If V is a future-pointing timelike four-vector, then there exists an inertial coordinate system in which it has components $(T, 0, 0, 0)$, where $T = \sqrt{g(V, V)}$.

   (ii) If V is a future-pointing null four-vector, then there exists an inertial coordinate system in which V has components $(1, 1, 0, 0)$.

   (iii) The sum of two future-pointing timelike four-vectors is future-pointing timelike.

   (iv) The sum of two future-pointing null four-vectors is future-pointing and either timelike or null. Under what condition is the sum null?

   (v) Every four-vector pseudo-orthogonal to a timelike vector is spacelike.

4. **A time-like inequality.** Let X and Y be future-pointing, timelike four-vectors, and let $Z = X + Y$. Show that

$$\sqrt{g(Z,Z)} \geqslant \sqrt{g(X,X)} + \sqrt{g(Y,Y)} \ .$$

When does equality hold? What is the analogous statement in Euclidean geometry?

Now consider two space-time events A and B separated by displacement vector Z, which is future-pointing timelike. One observer travels from A to B in a straight line at constant speed. A second observer travels from A to event C with displacement vector X from A in a straight line at constant speed, and then travels from C to B with displacement vector Y from C in a straight line at constant speed. Whose journey from A to B takes longer?

5. **Particle physics.** A particle of rest mass M and total energy E collides with a particle of rest mass m at rest. Show that the sum E' of the total energies of the two particles in the frame in which their center of mass is at rest is given by

$$E'^2 = (M^2 + m^2)c^4 + 2Emc^2 \ .$$

(Hint: let P and Q be the four-momenta of the two particles, and consider $g(P+Q, P+Q) = (E'/c^2)\, g(P+Q, V)$.)

The centre of mass is defined as the frame whose four-velocity V is proportional to the total four-momentum of the two particles.

6. **Photon scattering.** Suppose that two photons of energies E_1 and E_2 travel towards one another along the x-axis in a fixed ICS. Show that there can be no interaction in which the the outcome is a single photon. Argue that the same conclusion holds when the two photons don't necessarily collide head on, but instead collide at a general angle.

7. **Four-acceleration.** A particle travels along a straight line in space relative to a given ICS at a not-necessarily-constant speed. Show that

$$g(A,A) = -c^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 \ ,$$

where A is the four-acceleration, s measures proper time, and ϕ is the (instantaneous) rapidity.

8. **Constant acceleration motion.** Two rockets accelerating along the x-axis in opposite directions with constant acceleration a have worldlines whose coordinates in a fixed ICS are given by

$$x = -\frac{c^2 \cosh(as/c)}{a}, \quad t = \frac{c \sinh(as/c)}{a} \ ,$$

and

$$x = \frac{c^2 \cosh(as/c)}{a}, \quad t = \frac{c \sinh(as/c)}{a} \ ,$$

respectively.

Draw a space-time diagram showing the two worldlines. Show that the parameter s measures proper time along the worldlines.

Let $Z(s)$ denote the displacement four-vector from the event A at proper time $-s$ on the first worldline to the event B at proper time s on the second worldline. Show that

(i) $g(Z, Z)$ is independent of s.

(ii) Z is always pseudo-orthogonal to the four-velocity of the first rocket at A and to the four-velocity of the second rocket at B.

Deduce that observers in the two rockets reckon that A and B are simultaneous for every choice of s, and that they both think that the distance between A and B is independent of s. Thus the two rockets are always the same distance apart, according to the observers. Discuss this apparent absurdity.

Draw a picture of the Euclidean analogue of this situation.