B5.4/2016/Q1

(a) B/S – see online notes §3.2, Example 3. NB $c_m = \sqrt{\frac{T}{\sigma}}$

(b) B/S – see online notes §2.2 & 2.3

Note that pressure perturbation $p' = -\rho_0 \phi_t$ gives additional force per unit area acting on membrane, so must be added to RHS of membrane equation, i.e. $\sigma w_{tt} = T\nabla^2 w + p'$ on $z = d + w$ before linearizing onto $z = d$.

(c) Wave equation $\Rightarrow$ $-\frac{\omega^2}{c_0^2} = \underbrace{\frac{f'' + \frac{1}{r}f'}{f}}_{-\mu^2} + \underbrace{\frac{g''}{g}}_{-\frac{\lambda^2}{d^2}}$

$\phi_z = 0$ on $z = 0 \Rightarrow g(z) = A\cos\frac{\lambda z}{d}$

ϕ bdd at $r = 0$ & $\phi_t = 0$ on $r = a \Rightarrow f(r) = C J_0\left(\frac{j_i r}{a}\right)$, $\mu = \frac{j_i}{a}$ as in part (b).

Membrane equation $\Rightarrow -\sigma\omega^2 = -\mu^2 T + \rho_0 i\omega A\cos\lambda$

KBC on $z = 0$ $\Rightarrow -\frac{A\lambda}{d}\sin\lambda = -i\omega$

Eliminate A $\Rightarrow$ $\omega^2\left(\lambda\sin\lambda - \frac{\rho_0 d}{\sigma}\cos\lambda\right) = c_m^2\frac{j_i^2}{a^2}\lambda\sin\lambda$

$c_0^2\left(\frac{j_i^2}{a^2} + \frac{\lambda^2}{d^2}\right)$

For each i, find countably infinite set $\lambda = \lambda_{i,j}$.

(d) $\frac{\rho_0 d}{\sigma} \ll 1 \Rightarrow \lambda\sin\lambda\left(c_0^2\left(\frac{j_i^2}{a^2} + \frac{\lambda^2}{d^2}\right) - c_m^2\frac{j_i^2}{a^2}\right) = 0$

Either $\lambda = j\pi$ $(j \in \mathbb{Z}) \Rightarrow \omega^2 = c_0^2\left(\frac{j_i^2}{a^2} + \frac{j^2\pi^2}{d^2}\right)$

or $\omega^2 = c_0^2\left(\frac{j_i^2}{a^2} + \frac{\lambda^2}{d^2}\right) = c_m^2\frac{j_i^2}{a^2}$

<u>Physics</u>: gas light enough that normal modes are <u>uncoupled</u>

B5.4 / 2016 / Q2

(a) B – see online notes §2.3 & sheet 1, Q4

(b) B/S – see online notes §3.4 & sheet 3, Q2(ii)

Note $\hat{n}_{tt} + \omega(k)^2 \hat{n} = 0$ for $t > 0$ with $\hat{n} = \hat{f}$ and $\hat{n}_t = 0$ at $t = 0$, where $\omega(k) = \sqrt{gk\tanh(kh)}$

$\Rightarrow F(k) = \hat{f}(k)$.

(c) S/N – see online notes §3.5, "Group velocity" section.

Here $n(Vt, t) = I_+(t) + I_-(t)$

where $$I_\pm(t) = \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{4\pi} e^{i\psi_\pm(k)t}\, dk$$

$$\psi_\pm(t) = kV \mp \omega(k).$$

As explained in motivation to §3.5, dominant contributions to $I_\pm(t)$ as $t \to \infty$ are from neighborhoods of points of stationary phase $k = k_*^\pm$ where $\psi_\pm'(k_*^\pm) = 0$, i.e. from critical wavenumbers $k = k_*^\pm$ s.t. $\pm\omega'(k_*^\pm) = V$ or $\pm c_g(k_*^\pm) = V$.

Here, $c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh kh}$

$$c_g = \omega' = \frac{1}{2} c_p \left(1 + \frac{2kh}{\sinh(2kh)}\right)$$

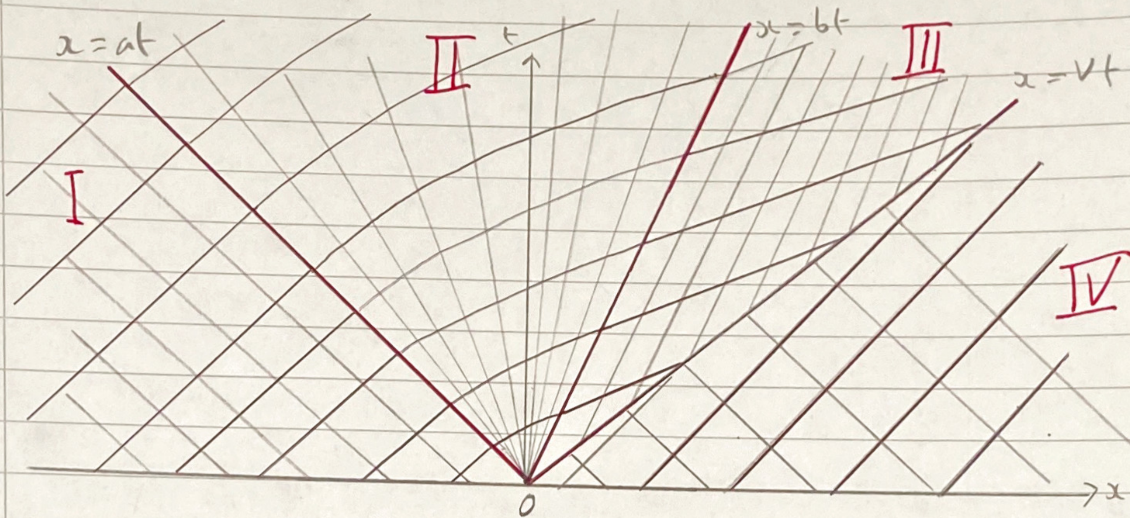
(i) $V \approx \sqrt{gh} \Rightarrow c_g \approx \sqrt{gh} \Rightarrow k_*^\pm h \ll 1 \Rightarrow c_p \sim c_g$
$\Rightarrow$ crests move with wave packet.

(ii) $V \ll \sqrt{gh} \Rightarrow c_g \ll \sqrt{gh} \Rightarrow k_*^\pm h \gg 1 \Rightarrow c_p \sim 2c_g$
$\Rightarrow$ crests move through wavepacket as in Fig 3.4(a)

B5.4|2016|Q3

(a) B – see online notes §4.3 & sheet 4, Q1

(b-c)



Region	C_- char^{cs}	C_+ char^{cs}
I	From $\{x<0, t=0\}$, so $u-2c=-2c_L$	From $\{x<0, t=0\}$, so $u+2c=2c_L$.
II	From $\{x=0, t=0\}$, with $u-c=\frac{x}{t}$	
III	From $\{x=Vt, t>0\}$, with $u-2c=u_--2c_-$	
IV	From $\{x>0, t=0\}$, so $u-2c=-2c_R$	From $\{x>0, t=0\}$, so $u+2c=2c_R$

NB: C_- char^{cs} everywhere straight; C_+ char^{cs} straight except in expansion fan region II.

C_+ char^{cs} carry info into LHS shock $\Rightarrow u_- + 2c_- = 2c_L$

$C_\pm$ char^{cs} carry info into RHS shock $\Rightarrow u_+ = 0,\ c_+ = c_R$

RHCs $\underset{(u_+=0)}{\Rightarrow}$ $V = -\dfrac{h_-u_-}{h_+-h_-} = \left(\dfrac{g(h_++h_-)h_-}{2h_+}\right)^{1/2}$

combo to get expressions for V in part (b); sketch e.g. $V(h_-, h_R)$ as a fn of h_- two ways to get uniqueness of h_- & hence u_- & V.

Into above $\Rightarrow$ $a=-c_L$, $b=2c_L-3c_-$; $c=c_L$, $u=0$ in I; $c=\frac{1}{3}(2c_L-\frac{x}{t})$, $u=\frac{2}{3}(c_L+\frac{x}{t})$ in II; $c=c_-$, $u=2(c_L-c_-)$ in III; and $c=c_R$, $u=0$ in IV.

B5.4/2017/Q1

(a) B – see online notes §2.3 & sheet 1, Q4

(b) Ansatz : $\eta = e^{-i\omega t} X(x)$, $\phi = e^{-i\omega t} X(x) Z(z)$

$$\nabla^2 \phi = 0 \implies \frac{X''}{X} = -\frac{Z''}{Z} = -k^2 \in \mathbb{R}_0^-$$

as trig solutions are required to satisfy $X'(0) = X'(a) = 0$.

$X'' + k^2 X = 0$ with $X'(0) = X'(a) = 0 \implies X = A\cos(k_n x)$, $k_n = \frac{n\pi}{a}$ with $n = 1, 2, \ldots$

$Z'' - k^2 Z = 0$ with $Z'(-h) = 0 \implies Z = B\cosh k(z+h)$

Linearized free surface conditions then give the required dispersion relation.

(c) New BC on $x = 0, a$ is $\phi_x = \mathrm{Re}(\varepsilon \Omega e^{-i\Omega t})$

Show problem for $\tilde{\phi}$ with $U = \varepsilon\Omega$ is the same as for ϕ except for linearized dynamic BC, which becomes

$$\tilde{\phi}_t + g\eta = i\varepsilon\Omega^2 (x - \tfrac{a}{2}) e^{-i\Omega t} = i\varepsilon\Omega^2 e^{-i\Omega t} \sum_{m=1}^{\infty} \frac{4\cos\left((2m-1)\frac{\pi x}{a}\right)}{a(2m-1)^2\left(\frac{\pi}{a}\right)^2}$$

by the hint. Hence superimpose solns in part (b):

$$\eta = \sum_{n=1}^{\infty} a_n \eta_n, \quad \tilde{\phi} = \sum_{n=1}^{\infty} a_n \phi_n, \text{ but with } \omega = \Omega.$$

This (formally) satisfies everything except the inhomogeneous dynamic BC above.

Substituting for η & $\tilde{\phi}$ & equating Fourier coeffs gives the a_n's and hence the series solution. Note that the $m=2$ mode dominates for Ω close to $\omega_{..}$

B5.4/2017/Q2

(a) B – see online notes §3.5.

(b,c) B/S – see sheet 3, Q1.

Note $\hat{\phi}(x,l,z) = \int_{-\infty}^{\infty} \phi e^{-ily}\,dy$, $\hat{\eta}(x,l) = \int_{-\infty}^{\infty} \eta e^{-ily}\,dy$

$\Rightarrow \hat{\phi}_{xx} + \hat{\phi}_{zz} = l^2 \hat{\phi}$ in $z<0$ with $\hat{\phi}_z = U\hat{\eta}_x$
and $U\hat{\phi}_x + g\hat{\eta} = 0$ on $z=0$; $\hat{\phi}_z \to 0$ as $z \to -\infty$
and $\hat{\eta} = \hat{\eta}_0$, $\hat{\eta}_x = 0$ at $x=0$.

Key step: seek sep. soln $\hat{\phi} = X(x)Z(z)$.

$$\Rightarrow \frac{X''}{X} + \frac{Z''}{Z} = l^2 \Rightarrow \frac{X''}{X} = \text{const.} = -k^2$$

for oscillatory solutions in the x-direction.

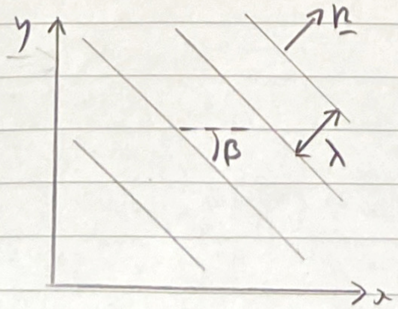
$$\Rightarrow \hat{\phi} = \left(A(l)\cos kx + B(l)\sin kx\right)e^{(k^2+l^2)^{1/2}z}$$

BCs on $z=0 \Rightarrow \hat{\eta} = \frac{kU}{g}(A\sin kx - B\cos kx)$, $(k^2+l^2)^{\frac{1}{2}} = \frac{U^2k^2}{g}$

ICs on $x=0 \Rightarrow A=0$, $\hat{\eta}_0 = -\frac{BkU}{g} \Rightarrow \hat{\eta} = \hat{\eta}_0 \cos(kx)$
and invert.

(d) Waves at edge of wake ($\lambda = \frac{1}{\sqrt{8}}$ say) are those with $s=2$, i.e. $l = \mp \frac{g}{U^2}\frac{\sqrt{3}}{2}$, $k = \frac{g}{U^2}\sqrt{\frac{3}{2}}$.

Hence, wavenumber vector $\underline{k} = (k,l) = \frac{g}{U^2}\sqrt{\frac{3}{2}}\left(1, \mp\frac{1}{\sqrt{2}}\right)$



Wavecrests are $\underline{k}\cdot(x,y) = \text{const}$ as illustrated for + sign.

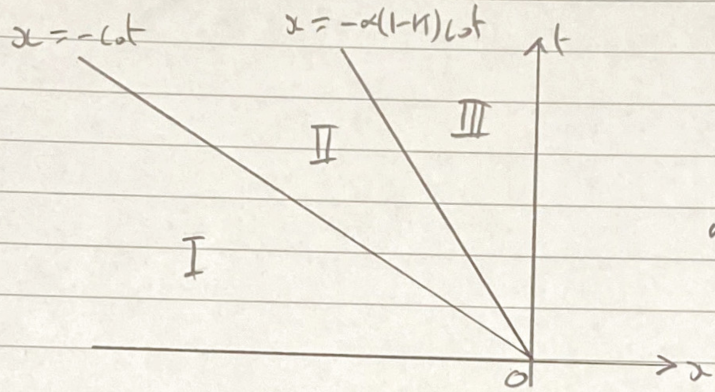
$$\Rightarrow \cos\beta = \frac{1}{\sqrt{3}},\quad \lambda = \frac{2\pi}{|\underline{k}|} = \frac{4\pi U^2}{3g}$$

B5.4/2017/Q3

(a) B – see online notes §1.9 and sheet 0, Q3.

(b) B – see online notes §4.2 & sheet 3, Q3

(c)



$\alpha = \dfrac{1}{1 + \frac{\gamma-1}{2}K}$

C_+ char^{cs} on which $u + 2c = 2c_0$ come everywhere from $\{x < 0, t = 0\}$.

I : C_- char^{cs} on which $u - \frac{2c}{\gamma-1} = -2c_0$ come from $\{x < 0, t = 0\}$ $\Rightarrow$ $u = 0$, $c = c_0$.

III : C_- char^{cs} on which $u - \frac{2c}{\gamma-1} = \text{const}$ come from $\{x = 0, t > 0\}$ on which $u = Kc$ $\Rightarrow$ $c = \alpha c_0$, $u = \alpha K c_0$

II : Needed because $-c_0 t < -\alpha(1-K)c_0 t < 0$ for $t > 0$ because $0 < \alpha < 1$ and $0 < K < 1$. Now C_- char^{cs} come from origin (expansion fan)

$$\Rightarrow \quad c = \frac{2}{\gamma+1}\left(c_0 - \frac{\gamma-1}{2}\frac{x}{t}\right), \quad u = \frac{2}{\gamma+1}\left(c_0 + \frac{x}{t}\right)$$

B5.4/2018/Q1

(a) B – see online notes §2.2

(b) S – see §2.2 "Waves due to a point source."

- $r\phi(r,t) = f(r)e^{-i\omega t}$

$\Rightarrow f'' + \frac{\omega^2}{c_0^2} f$ for $0 < r < a$

with $f(0) = 0$ (so $|\phi(0,t)| < \infty$) and $\frac{d}{dr}\left(\frac{f}{r}\right)\Big|_{r=a} = 0$

$\Rightarrow f(r) \propto \sin\left(\frac{\omega r}{c_0}\right)$, $\tan\left(\frac{\omega a}{c_0}\right) = \frac{\omega a}{c_0}$

(c) (i) Linearizing $\Rightarrow \phi_r = -i\omega a \varepsilon e^{-i\omega t}$ on $r = a$ (†)

(ii) $r\phi(r,t) = f(r)e^{-i\omega t} \Rightarrow f(r) = Ce^{i\omega r/c_0} + De^{-i\omega r/c_0}$ ($C, D \in \mathbb{C}$ arb)

$$\Rightarrow \phi = \underbrace{\frac{C}{r}e^{\frac{i\omega}{c_0}(r - c_0 t)}}_{\text{Outward}} + \underbrace{\frac{D}{r}e^{-\frac{i\omega}{c_0}(r + c_0 t)}}_{\text{Inward}}$$

Radiation condition $\Rightarrow D = 0$, so C given by (†)

(iii) For $\omega \neq \frac{c_0 \theta_n}{a}$ $\forall n$, $\phi = \frac{A}{r}\sin\left(\frac{\omega r}{c_0}\right)e^{-i\omega t}$

with A given by (†).

For $\omega = \frac{c_0\theta_m}{a}$ for some m, try instead

secular solution $\phi = \left(\frac{f(r)}{r} + \frac{t g(r)}{r}\right)e^{-i\omega t}$

$\Rightarrow g'' + \frac{\omega^2}{c_0^2}g = 0 \quad f'' + \frac{\omega^2}{c_0^2}f = -\frac{2i\omega}{c_0^2}g$

for $0 < r < a$, with $|\phi(0,t)| < \infty$ & (†) giving BCs.

B5.4/2018/Q2

(a) B – see online notes §3.5 & sheet 3, Q(i)

(b) Fourier transform in x $\Rightarrow$

$$(ik)^2\hat{\phi} + \hat{\phi}_{zz} = 0 \quad \text{for } z<0$$

$$\hat{\phi}_z = \hat{m}_t, \ \rho\hat{\phi}_t + B(ik)^4\hat{m} = 0 \ \text{on} \ z=0$$

$$\hat{\phi} \to 0 \ \text{as} \ z \to -\infty$$

$$\hat{m} = 0, \ \hat{m}_t = -\int_{-a}^{a} We^{-ikx}\,dx = -\frac{2W}{k}\sin(ka) \ \text{at } t=0.$$

Hence, $\hat{\phi} = A(k,t)e^{|k|z}$, so $|k|A = \hat{m}_t$, $\rho A_t = -Bk^4\hat{m}$

$$\Rightarrow \hat{m}_{tt} = -\frac{B}{\rho}k^4|k|\hat{m} \quad \text{for } t>0$$

$$\Rightarrow \hat{m} = -2W\frac{\sin(ka)}{k}\frac{\sin(\omega(k)t)}{\omega(k)}, \quad \omega(k) = \underbrace{\sqrt{\frac{B}{\rho}}}_{=\sigma}|k|^{\frac{5}{2}}$$

Finally, invert for answer.

(c) $m_t(Vt,t) = I_+(t) + I_-(t)$, $I_\pm(t) = \int_{-\infty}^{\infty} f(k)e^{i\psi_\pm(t)k}\,dk$

where $f(k) = -\frac{W\sin(ka)}{2\pi k}$, $\psi_\pm(t) = kV \mp \omega(k)$.

Now apply part (a): $\psi_\pm'(k_*^\pm) = 0$ iff $k_*^\pm = \pm\left(\frac{2V}{5\sigma}\right)^{2/3}$,

i.e. $\exists!$ point of stationary phase for $I_+(t)$ & $I_-(t)$.

Use part (a) with $\omega'(k) = \frac{5}{2}\sigma k|k|^{3/2}$, $\omega''(k) = \frac{15}{4}\sigma|k|^{\frac{1}{2}}$

gives answer with $A = -\frac{W\sin(k_*^\mp a)}{\pi k_*^\mp}\left(\frac{2\pi}{\frac{15}{4}\sigma(k_*^\mp)^{1/2}}\right)^{\frac{1}{2}}$

B5.4/2018/Q3

(a) B/S – see online notes §5.3 & sheet 4, Q3.

Rankine-Hugoniot conditions with $h_- = \alpha h_0$, $u_- = U$, $h_+ = h_0$ and $u_+ = 0$ imply

$$U = \left(1 - \tfrac{1}{\alpha}\right)V, \quad V = \left(\frac{\alpha(1+\alpha)gh_0}{2}\right)^{1/2},$$

taking the +ve root because $V > U > 0$.

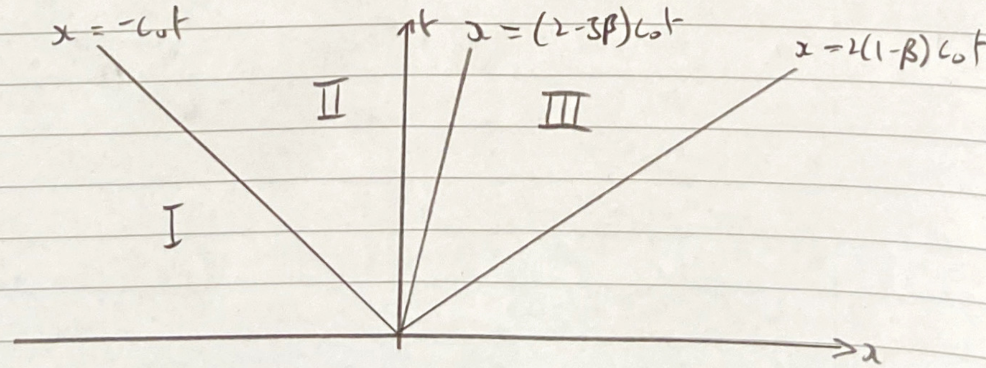
Eliminate V for equation for α.

(b) (i) B – see online notes §4.3 & sheet 4, Q1.

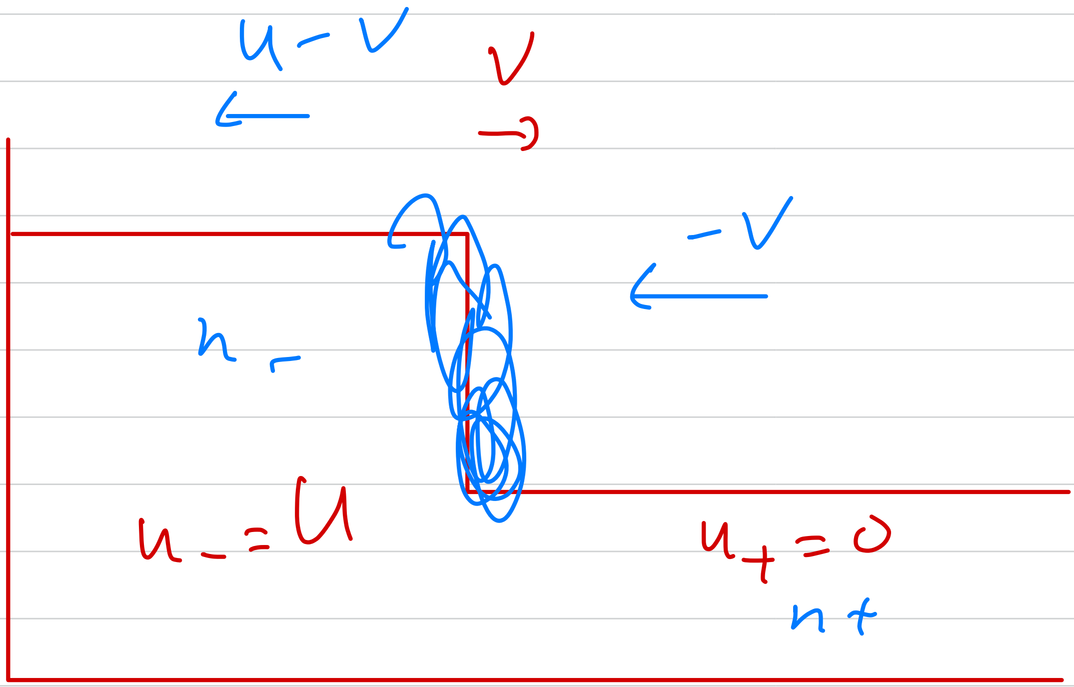
(ii) At $x = X(t)$, $u + 2c = 2c_0 = 2\sqrt{gh_0}$, $u = \dot{X}$ and $\dot{X} = \lambda h^2 = \lambda c^4/g$ by C_+ char^c, KBC & slip BC.

Together these imply $\frac{\lambda c^4}{g} + 2c = 2c_0$ at $x = X$, so that c is a constant, βc_0 say on dam, from which the results follow.

(iii)



C_+ char^{cs} come everywhere from $\{x < 0, t = 0\}$.
I: C_- char^{cs} from $\{x < 0, t = 0\} \Rightarrow u = 0, c = c_0$ there.
III: C_- char^{cs} from dam $\Rightarrow u = 2(1-\beta)c_0$, $c = \beta c_0$ there.
II: Needed because $0 < \beta < 1 \Rightarrow -c_0 t < (2-3\beta)c_0 t$.
II: C_- char^{cs} from origin (exp^n fan) $\Rightarrow u = \frac{2}{3}\left(c_0 + \frac{x}{t}\right)$
$c = \frac{1}{3}\left(2c_0 - \frac{x}{t}\right)$



$$h_- > h_+$$

2015, Q3(c)

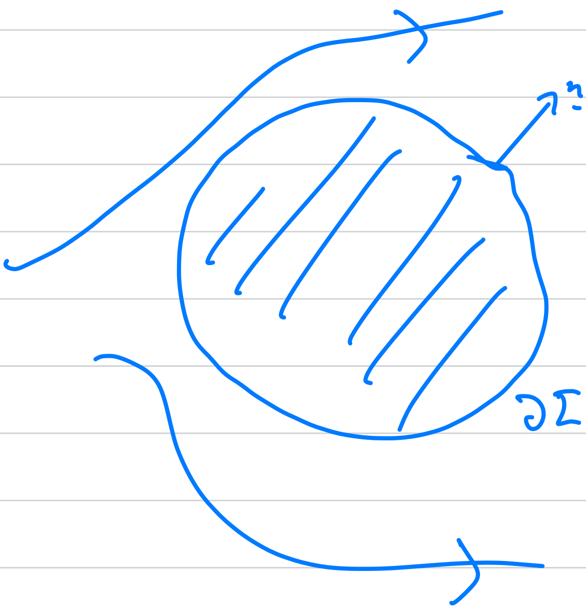
$$p = p_0 - \rho_0 U \phi_x$$

$$\Rightarrow \quad D = \int_{-a}^{a} f'(x)\left[\cancel{p_0} - \rho_0 U \phi_x(0-,x) + \cancel{p_0} - \rho_0 U \phi_x(0+,x)\right] dx$$

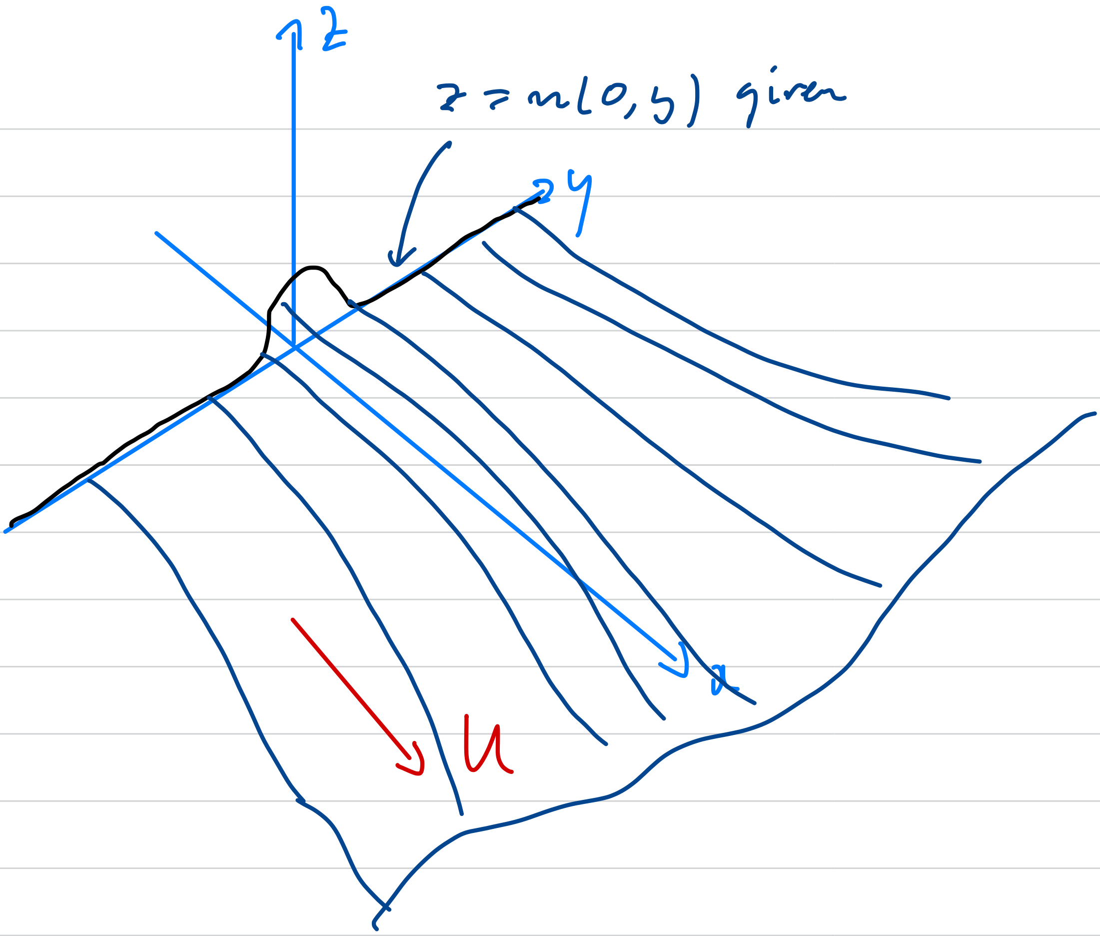
$$= \int_{-a}^{a} \frac{2\rho_0 U^2}{\beta} f'(x)^2 \, dx$$

$$= \frac{16}{3} \frac{\rho_0 U^2 b^2}{a\sqrt{\frac{U^2}{c_0^2} - 1}}$$

$$\frac{\partial D}{\partial U} = 0 \quad \Rightarrow \quad U = \sqrt{2}\, c_0$$



$$\underline{F} = \int_{\partial\Sigma} -p\,\underline{n}\, ds = (D, L)$$



z
$z = u(0,y)$ given
y
x
u