Special Relativity

Trinity Term 2018

Problem sheet 1

1. **Galilean group.** A Galilean transformation between the coordinate systems of two inertial frames R and R' is given by

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & H_{11} & H_{12} & H_{13} \\ v_2 & H_{21} & H_{22} & H_{23} \\ v_3 & H_{31} & H_{32} & H_{33} \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} ,$$

where $H \in \mathrm{SO}(3)$. ($\mathrm{SO}(3)$ is the group of 3×3 orthogonal matrices with positive determinant, so $H^{\mathrm{T}}H = I$ and $\det H = +1$).

   (a) Show that (v_1, v_2, v_3) are the components in frame R of the vector which gives the velocity of the origin of R'.

   (b) Show that the composition of two such transformations is again a Galilean transformation. Find the inverse of a Galilean transformation and show that it is again a Galilean transformation.

   (c) Let (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) be the coordinates relative to R of two events. Show that under Galilean transformations

      (A) The temporal separation $t_2 - t_1$ is invariant.

      (B) The spatial distance $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ is invariant provided that the events are simultaneous, *i.e.*, $t_2 = t_1$.

   Explain why statement (B) is not true for events that are not simultaneous. Show that, conversely, any coordinate transformation of the form

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} ,$$

   with properties (A) and (B), where M is a real, four-by-four matrix with positive determinant, is a Galilean transformation.

2. **Space-time diagrams.** Four ghosts travel in straight lines at different constant velocities across a flat (two-dimensional) field. Five of the six possible pairs of ghosts pass through each other at different times. Show that the sixth pair must also pass through each other at some point. It will be helpful to think in terms of a space-time diagram.

3. **Relativistic Doppler effect.** Two observers O and O' travel along the same straight line in space at constant speeds. The first sends out two light signals separated by an interval τ measured on O's clock. What is the interval between the times (according to O') when these signals are received at O' if both light signals are emitted

   (a) When O' is approaching O?

   (b) When O' is receding from O?

   If the light making up the signal has frequency ω as measured by O, what is the frequency as measured by O' in cases (a) and (b) respectively?

4. **Barn door paradox.** An pole-vaulter carrying a 15-foot-long pole runs with speed $\sqrt{3}c/2$ towards a barn that is 10 feet long. Exactly when the front of the pole reaches the far wall of the barn, a child standing by the barn door closes it. Explain, with the aid of a space-time diagram, how this is possible when in the athlete's coordinate system, the pole has a length 15 feet but the room is only 5 feet deep.

   If the athlete holds the pole by its back end, are they (i) outside the barn, (ii) at the barn door, or (iii) inside the barn door when they first feel the shock of the tip of the pole striking the far wall?

5. **Symmetries of the wave equation.** Consider the wave equation in one spatial dimension describing waves propagating at speed c,

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0 \ .$$

   (a) Consider the homogeneous Galilean transformation in one spatial dimension,

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} \ .$$

   Show that the wave equation is *not* invariant under this transformation, *i.e.*, there are solutions $\phi(x,t)$ equation that are not solutions of the corresponding equation in x' and t'.

   (b) Consider a transformation of the form

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} \ ,$$

   where p, q, r, and s are constant. Find necessary and sufficient conditions on p, q, r, and s to ensure that the wave equation in one spatial dimension is invariant, that is, that

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} - \frac{\partial^2\phi}{\partial x'^2} \ .$$

Assuming that p and s are both positive, show that we can rewrite this as the standard Lorentz transformation in one spatial dimension:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} , \quad \gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}} .$$

6. **Characterizing Lorentz transformation.** Consider a linear transformation

$$\begin{pmatrix} ct \\ x \end{pmatrix} = L \begin{pmatrix} ct' \\ x' \end{pmatrix} .$$

Show that L takes the form of the standard one-dimensional Lorentz transformation,

$$L = \gamma(v) \begin{pmatrix} 1 & v/c \\ v/c & 1 \end{pmatrix}$$

if and only if

(i) the top left entry in L is positive,

(ii) $\det L > 0$, and

(iii) L obeys the pseudo-orthogonality condition

$$L^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

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