Graph Theory Part A **Example Sheet 1** **Peter Keevash**

1. Show that any connected graph G has a vertex v such that $G - v$ is connected.

2. Show that a graph is minimally connected if and only if it is maximally acyclic.
('minimally P' means 'P holds but P doesn't hold if we delete any edge'; 'maximally P' means 'P holds but P doesn't hold if we add any new edge in the same set of vertices')

3. Let G be a connected graph. Show that any two paths of maximum length intersect.

4. Show that any tree on n vertices contains a path with k vertices or has at least n/k leaves.
Can you improve this statement?

5. Let G be a graph. Given $v \in V(G)$, the degree of v in G is $d_G(v) = |\{u \in V(G) : uv \in E(G)\}|$. Show that $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$. Deduce that G has an even number of vertices of odd degree.

6. Let G be a connected graph. Show that G has an Euler trail (i.e. a walk using each edge exactly once) if and only if there are at most two vertices with odd degree.

7. Let $d_1, \ldots, d_n$ be positive integers. Show that there is a tree on n vertices with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

8. Let G be a connected graph and $\{w(e) : e \in E(G)\}$ be distinct real numbers.
Show that (G, w) has a unique minimum cost spanning tree.

9. What is the maximum number of edges in a graph on n vertices with no triangle?