B5.4/2021/Q1

(a) Substitute $(\rho, \underline{u}, p) = (\rho_0, \underline{u}_0, p_0) + (\rho', \underline{u}', p')$ into (*) and linearize for small $\rho', \underline{u}', p'$ and Q.

$$\frac{D\rho}{Dt} = -\rho \nabla\cdot\underline{u} \Rightarrow \underline{\underline{\frac{d\rho'}{dt} \overset{①}{=} -\rho_0 \nabla\cdot\underline{u}'}} \text{ linearising, where } \underline{\underline{\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{u}_0\cdot\nabla}}.$$

$$\rho\frac{D\underline{u}}{Dt} = -\nabla p \Rightarrow \underline{\underline{\rho_0 \frac{d\underline{u}'}{dt} \overset{②}{=} -\nabla p'}} \text{ linearizing.}$$

$$\frac{\rho^{\gamma-1}}{(\gamma-1)}\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = Q \Rightarrow \frac{Dp}{Dt} - \frac{\gamma p}{\rho}\frac{D\rho}{Dt} = (\gamma-1)\rho Q$$

B/S4

$$\Rightarrow \underline{\underline{\frac{dp'}{dt} - c_0^2\frac{d\rho'}{dt} \overset{③}{=} (\gamma-1)\rho_0 Q}} \text{ linearizing, as } c_0^2 = \gamma p_0/\rho_0.$$

$$② \Rightarrow \frac{d}{dt}\nabla\wedge\underline{u}' = \nabla\wedge\frac{d\underline{u}'}{dt} = \nabla\wedge\left(-\frac{\nabla p'}{\rho_0}\right) = \underline{0}$$

$$\Rightarrow \nabla\wedge\underline{u}' = \underline{F}(\underline{x}-\underline{u}_0 t) = \underline{0} \text{ as } \underline{u}' = \underline{0} \text{ initially}$$

$\Rightarrow$ perturbed flow irrotational

By hint, there exists a disturbance potential ϕ s.t. $\underline{u}' \overset{④}{=} \nabla\phi$.

$② \Rightarrow \nabla\left(p' + \rho_0\frac{d\phi}{dt}\right) = \underline{0} \Rightarrow p' + \rho_0\frac{d\phi}{dt} = G(t) \overset{⑤}{=} 0$ wlog as ϕ is defined up to an arbitrary function of t.

$$\text{Hence, } \left(\frac{\partial}{\partial t} + \underline{u}_0\cdot\nabla\right)^2\phi = -\frac{1}{\rho_0}\frac{dp'}{dt} \qquad (\text{by } ⑤)$$

$$= -\frac{c_0^2}{\rho_0}\frac{d\rho'}{dt} - (\gamma-1)Q \qquad (\text{by } ③)$$

$$= c_0^2\nabla\cdot\underline{u}' - (\gamma-1)Q \qquad (\text{by } ①)$$

S5

$$= c_0^2\nabla^2\phi - (\gamma-1)Q \qquad (\text{by } ④) \quad \square$$

[9]

(b) Let $\phi = e^{-i\alpha c_0 t} f(x)\cos\left(\frac{\pi z}{h}\right)$ so that $\phi_z = 0$ on impermeable walls at $z = 0, h$ for $x > 0$.

Part (a) with $\underline{u}_0 \cdot \nabla = M c_0 \frac{\partial}{\partial x}$ and $Q = 0 \Rightarrow \left(-i\alpha c_0 + M c_0 \frac{d}{dx}\right)^2 f = c_0^2\left(\frac{d^2 f}{dx^2} - \frac{\pi^2 f}{h^2}\right)$

Let $f(x) = e^{\lambda x}$, then $(-i\alpha + M\lambda)^2 = \lambda^2 - \frac{\pi^2}{h^2}$, giving

$$(1 - M^2)\lambda^2 + 2i\alpha M\lambda + \alpha^2 - \frac{\pi^2}{h^2} = 0.$$

S3 Hence, $\lambda = \dfrac{-i\alpha M \pm \Delta^{1/2}}{1 - M^2}$, $\Delta = -\alpha^2 M^2 - (1 - M^2)\left(\alpha^2 - \frac{\pi^2}{h^2}\right) = (1 - M^2)\frac{\pi^2}{h^2} - \alpha^2$

Case (i) $\alpha h < \pi\sqrt{1 - M^2}$

Here $\Delta > 0$ and we take $\Delta^{1/2} > 0$, so that the solution

$$f(x) = A_1 \exp\left(\frac{-i\alpha M + \Delta^{1/2}}{1 - M^2} x\right) + A_2 \exp\left(\frac{-i\alpha M - \Delta^{1/2}}{1 - M^2}\right) \qquad (A_1, A_2 \in \mathbb{C})$$

is the linear superposition of exponentially growing or decaying modes as $x \to \infty$.

Hence impose ϕ bounded as $x \to \infty \Rightarrow |f(\infty)| < \infty \Rightarrow A_1 = 0$.

Then BC on $x = 0 \Rightarrow f'(0) = a \Rightarrow -\dfrac{(i\alpha M + \Delta^{1/2})}{1 - M^2} A_2 = a$, giving

S/N3 $$\phi = -\frac{(1 - M^2)a}{i\alpha M + \Delta^{1/2}} \exp\left(-\frac{i\alpha M + \Delta^{1/2}}{1 - M^2} x - i\alpha c_0 t\right)\cos\left(\frac{\pi z}{h}\right), \quad \Delta^{1/2} = \left[(1 - M^2)\frac{\pi^2}{h^2} - \alpha^2\right]^{\frac{1}{2}} > 0$$

Case (ii) $\alpha h > \pi$

Now $\Delta < 0$ so let $-\Delta = \varepsilon = \alpha^2 M^2 + (1 - M^2)\left(\alpha^2 - \frac{\pi^2}{h^2}\right) > \alpha^2 M^2$

$\Rightarrow \lambda = i\dfrac{\pm\varepsilon^{1/2} - \alpha M}{1 - M^2} = i\mu_\pm$ say, with $\mu_- < 0 < \mu_+$.

$\Rightarrow \phi = \left\{B_1 \exp(i(\mu_+ - \alpha c_0 t)) + B_2 \exp(i(\mu_- - \alpha c_0 t))\right\}\cos\left(\frac{\pi z}{h}\right)$ $(B_1, B_2 \in \mathbb{C})$

ϕ is linear superposition of right-travelling (B_1 term) and left-travelling (B_2 term) waves, so impose radiation condition as $x \to \infty \Rightarrow$ no incoming (left-travelling) waves $\Rightarrow B_2 = 0$.

Then BC at $x = 0 \Rightarrow i\mu_+ B_1 = a$, giving

S/N3
$$\underline{\underline{\phi = \frac{(1-M^2)a}{i(\varepsilon^{1/2} - \alpha M)} \exp\left(i\left(\frac{\varepsilon^{1/2} - \alpha M}{1-M^2} x - \alpha c_0 t\right)\right) \cos\left(\frac{\pi z}{h}\right)}}, \quad \underline{\underline{\varepsilon^{1/2} = \left[\alpha^2 M^2 + (1-M^2)\left(\alpha^2 - \frac{\pi^2}{h^2}\right)\right]^{\frac{1}{2}} > 0}}$$

[9]

(c)(i) $\phi = Be^{i(\underline{k}.\underline{x} - \omega t)} \Rightarrow \dfrac{d^2\phi}{dt^2} = (-i\omega + i\underline{u}_0.\underline{k})^2 \phi, \quad \nabla^2\phi = -|\underline{k}|^2\phi$

Part (a) $\Rightarrow -(\omega - \underline{u}_0 \cdot \underline{k})^2 B = -c_0^2 |\underline{k}|^2 B - (\gamma - 1)A$

S2
$$\underline{\underline{\Rightarrow \quad B = \frac{(\gamma-1)A}{(\omega - \underline{u}_0\cdot\underline{k})^2 - c_0^2|\underline{k}|^2} \quad \text{for } (\omega - \underline{u}_0 \cdot \underline{k})^2 \neq c_0^2 |\underline{k}|^2}}$$

(ii) If $(\omega - \underline{u}_0 \cdot \underline{k})^2 = c_0^2 |\underline{k}|^2$, try $\phi = Cte^{i(\underline{k}.\underline{x} - \omega t)}$ $\quad (C \in \mathbb{C})$

Then $\dfrac{d\phi}{dt} = (-i\omega + i\underline{u}_0.\underline{k}) Cte^{i(\underline{k}.\underline{x} - \omega t)} + Ce^{i(\underline{k}.\underline{x} - \omega t)}$

$\Rightarrow \dfrac{d^2\phi}{dt^2} = (-i\omega + i\underline{u}_0 \cdot \underline{k})^2 Cte^{i(\underline{k}.\underline{x} - \omega t)} + 2C(-i\omega + i\underline{u}_0.\underline{k})e^{i(\underline{k}.\underline{x} - \omega t)}$

while $\nabla^2 \phi = -|\underline{k}|^2 Cte^{i(\underline{k}.\underline{x} - \omega t)}$, so that part (a) gives

$$-\cancel{(\omega - \underline{u}_0 . \underline{k})^2 Ct} - 2Ci(\omega - \underline{u}_0.\underline{k}) = -\cancel{c_0^2 |\underline{k}|^2 Ct} - (\gamma - 1)A$$

Hence, $2iC(\omega - \underline{u}_0 \cdot \underline{k}) = (\gamma - 1)A$ giving a potential

N5
$$\underline{\underline{\phi = \frac{(\gamma-1)At}{2i(\omega - \underline{u}_0 \cdot \underline{k})} e^{i(\underline{k}.\underline{x} - \omega t)}}}$$

[7]

B5.4/2021/Q2

(a)(i) Let $\hat{\phi}(k,z,t) = \int_{-\infty}^{\infty} \phi(x,z,t)e^{-ikx}\,dx$, $\hat{m}(k,t) = \int_{-\infty}^{\infty} m(x,t)e^{-ikx}\,dk$.

$\nabla^2\phi = 0$ in $z<0$ $\Rightarrow$ $(ik)^2\hat{\phi} + \hat{\phi}_{zz} = 0$ in $z<0$ ①

BCs on $z=0$ $\Rightarrow$ $\hat{\phi}_z = \hat{m}_t$, $\sigma\hat{m}_{tt} + B(ik)^4\hat{m} = -\rho(\hat{\phi}_t + g\hat{m})$ on $z=0$ ②

BC at $z=-\infty$ $\Rightarrow$ $\hat{\phi} \to 0$ as $z \to -\infty$ ③

B/S3 ICs at $t=0$ $\Rightarrow$ $\hat{m}(k,0) = \hat{f}(k)$, $\hat{m}_t(k,0) = 0$ ④

① $\Rightarrow$ $\hat{\phi} = A(k,t)e^{|k|z} + B(k,t)e^{-|k|z}$ (A, B arb.)

③ $\Rightarrow$ $B = 0$

② $\Rightarrow$ $|k|A = \hat{m}_t$, $\sigma\hat{m}_{tt} - Bk^4\hat{m} = -\rho A_t - \rho g\hat{m}$

$\Rightarrow$ $\sigma|k|\hat{m}_{tt} + (\rho g + Bk^4)|k|\hat{m} = -\rho|k|A_t = -\rho\hat{m}_{tt}$

$\Rightarrow$ $\hat{m}_{tt} + \dfrac{(\rho g + Bk^4)|k|}{\rho + \sigma|k|}\hat{m} = 0$

$\Rightarrow$ $\hat{m}(k,t) = C(k)\cos\omega(k)t + D(k)\sin\omega(k)t$, $\underline{\underline{\omega(k) = \left[\dfrac{(\rho g + Bk^4)|k|}{\rho + \sigma|k|}\right]^{\frac{1}{2}}}}$

④ $\Rightarrow$ $C(k) = \hat{f}(k)$, $D(k) = 0$

Inverting, $m(x,t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{m}(k,t)e^{ikx}\,dk$

$= \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{f}(k)\cos\omega(k)t\, e^{ikx}\,dk$

$= \underline{\underline{\dfrac{1}{4\pi}\displaystyle\int_{-\infty}^{\infty} \hat{f}(k)\left\{e^{i(kx+\omega(k)t)} + e^{i(kx-\omega(k)t)}\right\}dk}}$

B/S5 as required, where $\omega(k)$ as given above. □

(ii) $\sigma = 0 \Rightarrow \omega(k) = \left[g|k| + \frac{B}{\rho}k^4|k|\right]^{1/2}$.

Phase speed $c_p(k) = \frac{\omega(k)}{k} = \frac{\left[g|k| + \frac{B}{\rho}k^4|k|\right]^{1/2}}{k}$.

Group velocity $c_g(k) = \frac{d\omega}{dk} = \frac{\left[g + \frac{B}{\rho}5k^4\right]\operatorname{sgn}(k)}{2\left[g|k| + \frac{B}{\rho}k^4|k|\right]^{1/2}}$

Individual waves move forwards through a wave packet moving with speed $c_g(k)$ iff $|c_g(k)| < |c_p(k)|$.

But $\frac{c_g(k)}{c_p(k)} = \frac{\left[g + 5\frac{B}{\rho}k^4\right]\operatorname{sgn}(k)}{2\left[g|k| + \frac{B}{\rho}k^4|k|\right]/k} = \frac{5g + 5\frac{B}{\rho}k^4 - 4g}{2(g + \frac{B}{\rho}k^4)} = \frac{5}{2} - \frac{2g}{g + \frac{B}{\rho}k^4}$,

so $|c_g(k)| < |c_p(k)| \Leftrightarrow \frac{5}{2} - \frac{2g}{g + \frac{B}{\rho}k^4} < 1 \Leftrightarrow \frac{3}{2}\left(g + \frac{B}{\rho}k^4\right) < 2g$

S/N3 $\Leftrightarrow \frac{B}{\rho}k^4 < \frac{g}{3} \Leftrightarrow |k| < k_c = \left(\frac{\rho g}{3B}\right)^{1/4}$ $\square$

[11]

(b)(i) $\eta = e^{-i\omega t}F(x) \Rightarrow -\sigma\omega^2 F - TF'' + BF'''' = 0$ for $0 < x < L$

with $F(0) = F''(0) = F(L) = F''(L) = 0$.

Let $f(x) = e^{\lambda x}$, then $B\lambda^4 - T\lambda^2 - \sigma\omega^2 = 0 \Rightarrow \lambda^2 = \frac{T \pm (T^2 + 4B\sigma\omega^2)^{1/2}}{2B}$

Hence, $\lambda = \pm\mu$ or $\pm i\varpi$, where $\mu > 0$ and $\varpi > 0$ are defined by

$$\mu = \left\{\frac{T + (T^2 + 4B\sigma\omega^2)^{1/2}}{2B}\right\}^{1/2}, \quad \varpi = \left\{\frac{(T^2 + 4B\sigma\omega^2)^{1/2} - T}{2B}\right\}^{1/2}.$$

S3 Thus, $F(x) = A_1\cosh\mu x + A_2\sinh\mu x + A_3\cos\varpi x + A_4\sin\varpi x$ (with $A_i \in \mathbb{C}$)

$$\left.\begin{aligned} F(0) = 0 &\Rightarrow A_1 + A_3 = 0 \\ F''(0) = 0 &\Rightarrow \mu^2 A_1 - \varpi^2 A_3 = 0 \end{aligned}\right\} \Rightarrow A_1 = A_3 = 0$$

$$\left.\begin{aligned} F(L) = 0 &\Rightarrow A_2\sinh\mu L + A_4\sin\varpi L = 0 \\ F''(L) = 0 &\Rightarrow \mu^2 A_2\sinh\mu L - \varpi^2 A_4\sin\varpi L = 0 \end{aligned}\right\} \Rightarrow A_2 = 0,\ A_4\sin\varpi L = 0$$

Since $A_1 = A_2 = A_3 = 0$, we need $A_4 \neq 0$ for a nontrivial solution, so $\sin \omega L = 0 \Rightarrow \omega L = n\pi$, $n \in \mathbb{Z}^+$ wlog.

Hence, normal modes and corresponding natural frequencies ω are given for positive integers n by

S4 $$\eta = A_4 \sin\left(\frac{n\pi x}{L}\right) \ (A_4 \in \mathbb{C}), \quad \omega^2 = \frac{B}{\sigma}(\omega)^4 - \frac{T}{\sigma}(\omega)^2 = \frac{B}{\sigma}\left(\frac{n\pi}{L}\right)^4 + \frac{T}{\sigma}\left(\frac{n\pi}{L}\right)^2$$

(ii) $T = 0 \Rightarrow \omega^2 = \Omega^2$ for $n = 1$, so try $\phi = e^{-i\Omega t}\left(F(x) + tG(x)\right)$

$$\sigma \eta_{tt} + B\eta_{xxxx} = 0 \Rightarrow -\Omega^2 F - 2i\Omega G - \Omega^2 tG + \frac{B}{\sigma}\left(F'''' + tG''''\right) = 0$$

$$\Rightarrow F'''' - \frac{\pi^4}{L^4}F = \frac{2i\sigma\Omega}{B}G, \quad G'''' - \frac{\pi^4}{L^4}G = 0 \text{ for } 0 < x < L$$

S/N3 with $F(0) = F''(0) = F(L) = 0$, $F''(L) = A$ and $G(0) = G''(0) = G(L) = G''(L) = 0$.

Since $G = B_0 \sin\left(\frac{\pi x}{L}\right)$ $(B_0 \in \mathbb{C}$ arb$)$, $F'''' - \frac{\pi^4}{L^2}F = \frac{2i\sigma\Omega}{B} B_0 \sin\left(\frac{\pi x}{L}\right)$

Particular soln $F(x) = B_1 x \cos\left(\frac{\pi x}{L}\right) \Rightarrow F' = B_1 \cos\left(\frac{\pi x}{L}\right) - B_1 x\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{L}\right)$

$$F'' = -2B_1\left(\frac{\pi}{L}\right)\sin\left(\frac{\pi x}{L}\right) - B_1 x\left(\frac{\pi}{L}\right)^2\cos\left(\frac{\pi x}{L}\right)$$
$$F''' = -3B_1\left(\frac{\pi}{L}\right)^2\cos\left(\frac{\pi x}{L}\right) + B_1 x\left(\frac{\pi}{L}\right)^3\sin\left(\frac{\pi x}{L}\right)$$
$$F'''' = 4B_1\left(\frac{\pi}{L}\right)^3\sin\left(\frac{\pi x}{L}\right) + B_1 x\left(\frac{\pi}{L}\right)^4\cos\left(\frac{\pi x}{L}\right)$$

Hence, ODE for $F \Rightarrow 4B_1\left(\frac{\pi}{L}\right)^3 = \frac{2i\sigma\Omega}{B}B_0 \Rightarrow B_0 = \frac{2B}{i\sigma\Omega}\left(\frac{\pi}{L}\right)^3 B_1$

General solution $F(x) = B_1 x\cos\left(\frac{\pi x}{L}\right) + C_1\cosh\frac{\pi x}{L} + C_2\sinh\frac{\pi x}{L} + C_3\cos\frac{\pi x}{L} + C_4\sin\frac{\pi x}{L}$ $(C_i \in \mathbb{C}$ arb.$)$

$$\left.\begin{aligned} F(0) &= 0 \Rightarrow C_1 + C_3 = 0 \\ F''(0) &= 0 \Rightarrow \left(\tfrac{\pi}{L}\right)^2(C_1 - C_3) = 0 \end{aligned}\right\} \Rightarrow C_1 = C_3 = 0$$

$$\left.\begin{aligned} F(L) &= 0 \Rightarrow -B_1 L + C_2\sinh\pi = 0 \\ F''(L) &= A \Rightarrow B_1 L\left(\tfrac{\pi}{L}\right)^2 + C_2\left(\tfrac{\pi}{L}\right)^2\sinh\pi = A \end{aligned}\right\} \Rightarrow \begin{aligned} B_1 &= LA/2\pi^2 \\ C_2 &= L^2A/2\pi^2\sinh\pi \end{aligned}$$

N4 [14] $$\text{Hence, } \eta = e^{-i\Omega t}\left\{\frac{LA}{2\pi^2}\left(x\cos\left(\frac{\pi x}{L}\right) + \frac{L\sinh\left(\frac{\pi x}{L}\right)}{\sinh\pi}\right) + C_4\sin\left(\frac{\pi x}{L}\right) + t\,\frac{\pi BA}{i\sigma\Omega L^2}\sin\left(\frac{\pi x}{L}\right)\right\}$$

B5.4/2021/Q3

(a) The Rankine-Hugoniot conditions imply that

$$\rho_0(0-V) = \rho_-(u_--V), \qquad (1)$$
$$p_0 + \rho_0(0-V)^2 = p_- + \rho_-(u_--V)^2, \qquad (2)$$
$$\tfrac{1}{2}(0-V)^2 + 2p_0/\rho_0 = \tfrac{1}{2}(u_--V)^2 + \frac{2p_-}{\rho_-}, \qquad (3)$$

while the BC on the piston gives

B2

$$\rho_-(u_--U) = -\lambda(p_--p_0). \qquad (4)$$

(1)-(4) are the 4 equations for the 4 unknowns ρ_-, p_-, u_- and V.

(1) & (2) $\Rightarrow$ $p_- = p_0 + \rho_0 V^2 + \rho_0 V(u_--V) = p_0 + \rho_0 u_- V$ (5)

(1) & (5) in (3) $\Rightarrow$ $\tfrac{1}{2}(u_--V)^2 + 2(p_0+\rho_0 u_- V)\left(\frac{u_--V}{-\rho_0 V}\right) = \tfrac{1}{2}V^2 + \frac{2p_0}{\rho_0}$

$$\Rightarrow \tfrac{1}{2}u_-^2 - u_-V - \frac{2p_0}{\rho_0}\left(\frac{u_--V}{V}+1\right) - 2u_-(u_--V) = 0$$

$$\Rightarrow -\tfrac{3}{2}u_-^2 + u_-V - \frac{2p_0}{\rho_0}\frac{u_-}{V} = 0$$

S3

$\Rightarrow u_- = \dfrac{2(V^2-c_0^2)}{3V}$ (6), where $c_0 = \sqrt{\frac{2p_0}{\rho_0}}$ as $u_- \neq 0$ for a shock by (5)

(1) & (5) in (4) $\Rightarrow$ $\dfrac{-\rho_0 V}{u_--V}(u_--U) = -\lambda\rho_0 u_- V$ ($u_- \neq V$ for a shock by (1))

$\Rightarrow u_- - U = \lambda u_-(u_--V)$ ($\rho_0 V \neq 0$)

$$\Rightarrow \lambda u_-^2 - (1+\lambda V)u_- + U = 0$$

S3

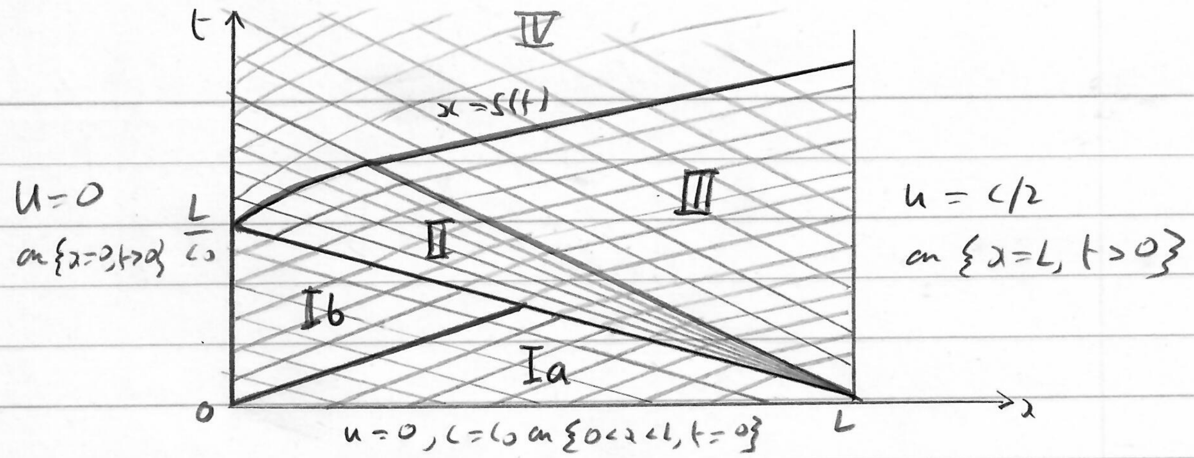
$$\Rightarrow u_- = \frac{1+\lambda V \pm [(1+\lambda V)^2 - 4\lambda U]^{1/2}}{2\lambda} \qquad (7)$$

N2

[10]

But (4) $\Rightarrow$ $u_- = U$ for $\lambda = 0$, so for λ suff. small need $-$ root in (7) and then (6) & (7) $\Rightarrow$ $\underline{\underline{\dfrac{4\lambda(V^2-c_0^2)}{3V} = 1+\lambda V - [(1+\lambda V)^2 - 4\lambda U]^{1/2}}}$ $\square$

(b)(i)



<u>Region Ia</u>: Where $\pm$ char^{cs} from $\{0<x<L, t=0\}$ intersect, $u \pm 2c = \pm 2c_0$ $\Rightarrow u = 0, c = c_0 \Rightarrow$ they are straight with $\frac{dx}{dt} = \pm c_0 \Rightarrow$ they
B2 map out $c_0 t < x < L - c_0 t$ for $0 < t < L/2c_0$.

<u>Region Ib</u>: $-$char^{cs} from $\{0<x<L, t>0\}$, so $u - 2c = -2c_0$, so on piston at $x = 0$ have $c = c_0$ because $u = 0$ there. Hence, on $+$char^{cs} from $x = 0$ have $u + 2c = 2c_0$. Where these $\pm$ char^{cs} intersect have $u = 0, c = c_0 \Rightarrow$ they are straight with $\frac{dx}{dt} = \pm c_0$ $\Rightarrow$ they map out $x < c_0 t$, $x < L - c_0 t$ for $0 < t < L/c_0$.

B/S3 Combo $\Rightarrow u = 0, c = c_0$ in regions Ia & Ib where $0 < x < L - c_0 t, 0 < t < \frac{L}{c_0}$.

<u>Region III</u>: $+$char^{cs} from $\{0<x<L, t=0\} \cup \{x=0, 0<t<L/c_0\}$ so $u + 2c = 2c_0$, so on piston at $x = L$ where $u = c/2$, have $c = \frac{4}{5}c_0, u = \frac{2}{5}c_0$. Hence, on $-$char^{cs} from $x = L$, have $u - 2c = -\frac{6}{5}c_0$. Where these $\pm$ char^{cs} intersect, have $u = \frac{2}{5}c_0$, $c = \frac{4}{5}c_0 \Rightarrow$ they are straight with $\frac{dx}{dt} = (\frac{2}{5} \pm \frac{4}{5})c_0$. Region III bounded ~~as~~ below by first $-$chars originating from $(x,t) = (L,0)$, namely $x = L - \frac{2}{5}c_0 t$. Hence, $u = \frac{2}{5}c_0$, $c = \frac{4}{5}c_0$ in $L - \frac{2}{5}c_0 t < x < L$ for $0 < t < L/c_0$ since $+$ chars from
S3 $(x,t) = (0, L/c_0)$ ($x = s(t)$ in diagram) lies in $t > L/c_0$ where $u, c \geq 0$.

<u>Region II</u>: $+$char^{cs} still from $\{0<x<L, t=0\} \cup \{x=0, 0<t<L/c_0\}$, so $u + 2c = 2c_0$. On a $-$char^{c}, $u - 2c = R_- = $ const, so where it intersects this family of $+$char^{cs} have u & c constant $\Rightarrow -$char^{c} is straight. To avoid it crossing other $-$char^{cs} in regions I, II and III, it must originate from $(x,t) = (L, 0)$, i.e. an expansion fan.

Since $-$ char$^{\underline{c}}$ is straight with $\frac{dx}{dt} = u - c$ and passes through $(x,t) = (L,0)$, have $u - c = \frac{dx}{dt} = \frac{x-L}{t}$. Solving $u + 2c = 2c_0$ and $u - c = \frac{x-L}{t}$ gives $u = \frac{2}{3}\left(c_0 + \frac{x-L}{t}\right)$, $c = \frac{1}{3}\left(2c_0 - \frac{x-L}{t}\right)$ between regions I and III for $x > s(t)$, so certainly for

S3 $L - c_0 t < x < L - \frac{2}{5}c_0 t$, $0 < t < L/c_0$.

Note u and c cts, so obtain in particular, for $0 < x < \frac{L}{c}$,

$$u = \begin{cases} 0 & \text{for } 0 \le x \le L - c_0 t, \\ \frac{2}{3}\left(c_0 + \frac{x-L}{t}\right) & \text{for } L - c_0 t \le x \le L - \frac{2}{5}c_0 t, \\ \frac{2}{5}c_0 & \text{for } L - \frac{2}{5}c_0 t \le x \le L. \end{cases}$$

(ii) See sketch above of characteristic diagram.

Region II where $+$ char$^{\underline{s}}$ from $\{0 < x < L, t = 0\} \cup \{x = 0, 0 < t < \frac{L}{c_0}\}$ intersect $-$ char$^{\underline{s}}$ from $(x,t) = (L,0)$, so bdd above by $+$ char$^{\underline{c}}$ $x = s(t)$ from $(x,t) = (0, L/c_0)$. To find it solve $\frac{ds}{dt} = u + c = \frac{4}{3}c_0 + \frac{s-L}{3t}$ with $s = 0$ at $t = L/c_0$ until $x = s(t)$ crosses $x = L - \frac{2}{5}c_0 t$, i.e. until $t = t^*$ when $s(t^*) = L - \frac{2}{5}c_0 t^*$.

Similarly region III bdd above by same $+$ char$^{\underline{c}}$, which is now determined by solving $\frac{ds}{dt} = u + c = \frac{6}{5}c_0$ with $s = L - \frac{2}{5}c_0 t^*$ at $t = t^*$ until $x = s(t)$ crosses $x = L$.

Combo $\Rightarrow$ sol$^{\underline{n}}$ in part (b)(i) still holds for $s(t) \le x \le L$, and it does not hold for $x < s(t)$ because $c \ne c_0$ on $x = 0$ for $t > L/c_0$ because $c \ne c_0$ on $-$ char$^{\underline{s}}$ in expansion fan in region II, i.e. solution no longer a simple wave with $u + 2c = 2c_0$

N4 in region IV.

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