1. Show that a graph is bipartite if and only if it does not have a cycle of odd length.

2. Show that any graph has a bipartite subgraph containing at least half of its edges.

3. Let G be a bipartite graph in which all vertices have the same degree. Show that G has a perfect matching.

4. Let G be a graph in which the degree of every vertex is in $\{1, \ldots, k\}$. Show that G has a matching with at least |V(G)|/2k edges.

5. Let M be a square matrix. Say that M is doubly stochastic if it has non-negative entries, every row sum is 1 and every column sum is 1. Say that M is a permutation matrix if every row or column has one entry equal to 1 and all other entries equal to 0. Show that any doubly stochastic matrix M can be written as $M = \sum_{i=1}^{k} c_i P_i$ for some k, where P_i is a permutation matrix and $c_i > 0$ for $1 \le i \le k$.

6. Show that a complete graph with $2^t + 1$ vertices cannot be expressed as the union of t bipartite graphs. Deduce that in any set of $2^t + 1$ points in the plane we can find a triangle with an angle of at least $(1 - 1/t)\pi$ radians.

7. A travelling salesman is required to follow a route that visits every city exactly once and returns to the starting point. How can (s)he find the best route? In the language of graph theory, we have a complete graph on n vertices, a positive weight w(xy) 'the distance' assigned to each pair of vertices xy, and we are required to find a minimum weight 'Hamilton cycle', i.e. a cycle that uses all n vertices. We say that an algorithm is an r-approximation algorithm if it always finds some Hamilton cycle with weight that is at most r times as large as the minimum possible.

Suppose that you have an efficient computer program that implements an r-approximation algorithm for the travelling salesman problem (TSP). Describe how you could use this program to test whether any given graph contains a Hamilton cycle. (It is known that the latter task is hard unless 'P=NP', so the conclusion of this exercise is that there is unlikely to be an efficient approximation algorithm for TSP.)

8. (continuation of the previous exercise)

Show that the following algorithm gives a 2-approximation for TSP under the additional assumption that the distances satisfy the triangle inequality $w(xz) \leq w(xy) + w(yz)$ for all x, y, z.

Insertion Algorithm. Start with any vertex v_1 . Suppose at step $1 \le t < n$ we have a sequence of distinct vertices $s_t = (v_1, \ldots, v_t)$. Pick $1 \le i \le t$ and $u \notin \{v_1, \ldots, v_t\}$ to minimise $w(v_i u)$. Define s_{t+1} by inserting u into s_t after v_i . Output the Hamilton cycle defined by s_n .

9. Suppose that n students want to share m identical cakes equally, where m and n are coprime. What is the minimum number of cuts that they need to make?