The simplest example is just given by taking $L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ for motion in free space without any forces. The Euler-Lagrange equations are just

$$\ddot{x} = \ddot{y} = \ddot{z} = 0\,, \tag{44}$$

i.e. Newton's laws of motion for a free particle.

The next simplest example arises from $L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - m\overset{\psi}{\cancel{\phi}}(x, y, z)$ for motion in free space subject only to a conservative force with potential $\cancel{\phi}$ ψ (typically, Newtonian gravity.) The Euler-Lagrange equations then become

$$\ddot{x} = -\frac{\partial \cancel{\phi}\psi}{\partial x}, \ddot{y} = -\frac{\partial \cancel{\phi}\psi}{\partial y}, \ddot{z} = -\frac{\partial \cancel{\phi}\psi}{\partial z}, \tag{45}$$

as required.

The value of the reformulation as a stationary integral emerges more clearly if we make a *change of coordinates.* For orbit problems, with $\cancel{\phi}\psi = -k/r$, the use of Cartesian x, y, z is correct but not very helpful. Since the Lagrangian formalism does not mind which coordinates we use, let's use spherical polars instead. Then

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) + \frac{km}{r}\,. \tag{46}$$

The θ-equation is:

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) - r^2 \sin\theta\cos\theta\dot{\phi}^2 = 0\,, \tag{47}$$

which is solved by $\theta \equiv \pi/2$, i.e. by paths always in the equatorial plane. Restricting our attention to such paths, the remaining equations become

$$\ddot{r} - r\dot{\phi}^2 + \frac{k}{r^2} = 0\,, \tag{48}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\phi}) = 0\,, \tag{49}$$

which we can recognise as the equations obtained by a longer argument in the Prelims treatment. The ϕ-equation obviously integrates to

$$r^2\dot{\phi} = h\,. \tag{50}$$

It is very important to note that the simplicity of this step arises directly from the fact that ϕ never appears in L; it is an ignorable coordinate. So in the Lagrangian formulation, the conservation of angular momentum is an *immediate* consequence.