**Introduction to Manifolds**

Trinity Term 2019

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**Example sheet 2**

1. (a) Show that there exists a real-valued $C^1$ function $g$ defined on a neighbourhood of the origin in $\mathbb{R}$ such that

$$g(x) = (g(x))^3 + 2e^{g(x)} \sin x.$$

   (b) Show that the equations

$$\begin{aligned} e^x + e^{2y} + e^{3u} + e^{4v} &= 4 \\ e^x + e^y + e^u + e^v &= 4 \end{aligned}$$

   can be solved for $u, v$ in terms of $x, y$ near the origin.

2. By considering the function defined by

$$f(x) = \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) \text{ for } x \neq 0 \text{ and } f(0) = 0,$$

   show that the $C^1$ hypothesis cannot be removed from the statement of the Inverse Function Theorem.

3. Deduce the Inverse Function Theorem from the Implicit Function Theorem.

   (Hint: consider functions like $(x, y) \mapsto f(x) - y$).

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$ function.

   (a) Show that the graph of $f$

$$\{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}$$

   is a 2-dimensional submanifold of $\mathbb{R}^3$.

   (b) Identify the normal space to $M$ at a point $(x, y, f(x, y))$ and give a basis for the tangent space at that point.

5. For which values of $c$ does the equation

$$x^2 + y^2 - z^2 = c$$

   define a 2-dimensional submanifold of $\mathbb{R}^3$?

   Describe the loci defined by the above equation, paying particular attention to any values for which the locus is not a manifold.

6. Find the maximum value of

$$g(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i$$

subject to the constraint $\sum_{i=1}^{n} x_i = 1$ and the condition that the $x_i$ are nonnegative. Deduce the arithmetic mean/geometric mean inequality

$$\left( \prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$$

for nonnegative reals $a_1, \ldots, a_n$.