Solutions

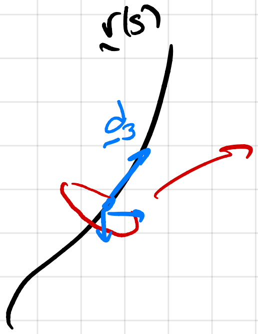
① (a)
$$\begin{cases} \underset{\sim}{R}' = \underset{\sim}{T} \\ \underset{\sim}{T}' = \kappa \underset{\sim}{N} \\ \underset{\sim}{N}' = -\kappa \underset{\sim}{T} + \tau \underset{\sim}{B} \\ \underset{\sim}{B}' = -\tau \underset{\sim}{N} \end{cases}$$

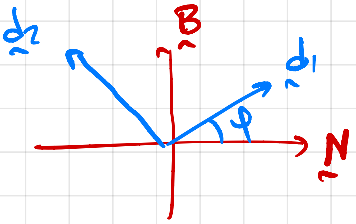
[3 marks]

(i) $\{\underset{\sim}{T}, \underset{\sim}{N}, \underset{\sim}{B}\}$ frame is defined only by the space curve $\underset{\sim}{r}$. the material frame $\{\underset{\sim}{d}_1, \underset{\sim}{d}_2, \underset{\sim}{d}_3\}$ also encodes the orientation of the cross-section, and will differ if there is twist.

(ii) $\underset{\sim}{d}_3 = \underset{\sim}{T}$

$\Rightarrow$ $\underset{\sim}{d}_1, \underset{\sim}{d}_2$ are co-planar w/ $\underset{\sim}{N}, \underset{\sim}{B}$,





ie the material frame for the cross-section, $\{\underset{\sim}{d}_1, \underset{\sim}{d}_2\}$ will satisfy

$$\underset{\sim}{d}_1 = \cos\varphi \, \underset{\sim}{N} + \sin\varphi \, \underset{\sim}{B}$$
$$\underset{\sim}{d}_2 = -\sin\varphi \, \underset{\sim}{N} + \cos\varphi \, \underset{\sim}{B}$$

where $\varphi(s)$ is the register angle

[Not needed]

[2]

The curvature vector $\underset{\sim}{u}$ satisfies

$$\underset{\sim}{d}_i' = \underset{\sim}{u} \wedge \underset{\sim}{d}_i \qquad \left(' = \frac{d}{ds} \right),$$ that is

$$\underset{\sim}{d}_1' = u_3 \underset{\sim}{d}_2 - u_2 \underset{\sim}{d}_3 \qquad (1)$$
$$\underset{\sim}{d}_2' = u_1 \underset{\sim}{d}_3 - u_3 \underset{\sim}{d}_1 \qquad (2)$$
$$\underset{\sim}{d}_3' = u_2 \underset{\sim}{d}_1 - u_1 \underset{\sim}{d}_2 \qquad (3)$$

[2]

$$(1) \Rightarrow \cos\varphi \underset{\sim}{N}' - \sin\varphi\, \varphi' \underset{\sim}{N} + \sin\varphi \underset{\sim}{B}' + \cos\varphi\, \varphi' \underset{\sim}{B}$$

$$= u_3 \left(-\sin\varphi \underset{\sim}{N} + \cos\varphi \underset{\sim}{B}\right) - u_2 \underset{\sim}{T}$$

using the Frenet relations and equating components:

$\underset{\sim}{T}$: $-\kappa\cos\varphi = -u_2 \Rightarrow \boxed{u_2 = \kappa\cos\varphi}$ [1]

$\underset{\sim}{N}$: $-\sin\varphi\,\varphi' - \tau\sin\varphi = -u_3\sin\varphi \Rightarrow \boxed{u_3 = \varphi' + \tau}$ [1]

We can obtain u_1 via $(3)\cdot\underset{\sim}{d}_2 \rightarrow u_1 = -\underset{\sim}{d}_3{}' \cdot \underset{\sim}{d}_2 = -\kappa \underset{\sim}{N} \cdot \left(-\sin\varphi \underset{\sim}{N} + \cos\varphi \underset{\sim}{B}\right)$

$$\Rightarrow \boxed{u_1 = \kappa\sin\varphi}$$ [1]

(b)

(i) The kinematic equations $\left\{ \begin{array}{l} \underset{\sim}{r}'(s) = \underset{\sim}{d}_3 , \\ \underset{\sim}{d}_i'(s) = \underset{\sim}{u} \wedge \underset{\sim}{d}_i \end{array} \right.$

w/ $\underline{u} = \hat{\kappa}\, \underset{\sim}{d}_2 + \hat{\tau}\, \underset{\sim}{d}_3$, read $\left\{ \begin{array}{l} \underset{\sim}{d}_1' = \hat{\tau}\, \underset{\sim}{d}_2 - \hat{\kappa}\, \underset{\sim}{d}_3 \\ \underset{\sim}{d}_2' = -\hat{\tau}\, \underset{\sim}{d}_1 \\ \underset{\sim}{d}_3' = \hat{\kappa}\, \underset{\sim}{d}_1 \end{array} \right.$ [1]

$$\underset{\sim}{r} = \begin{pmatrix} a \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ a \sin(\cdot) \\ \frac{bs}{\sqrt{\cdot}} \end{pmatrix} \Rightarrow \underline{r}' = \underset{\sim}{d}_3 = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -a \sin(\cdot) \\ a \cos(\cdot) \\ b \end{pmatrix}$$

Then $\underset{\sim}{d}_3' = \frac{-a}{a^2+b^2} \begin{pmatrix} \cos(\cdot) \\ \sin(\cdot) \\ 0 \end{pmatrix} = \hat{\kappa}\, \underset{\sim}{d}_1 \Rightarrow \boxed{\hat{\kappa} = \frac{a}{a^2+b^2}} , \underset{\sim}{d}_1 = - \begin{pmatrix} \cos() \\ \sin() \\ 0 \end{pmatrix}$

And $\underset{\sim}{d}_1' = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} \sin(\ ) \\ -\cos(\ ) \\ 0 \end{pmatrix} = -\hat{\kappa}\, \underset{\sim}{d}_3 + \hat{\tau}\, \underset{\sim}{d}_2$ [3]

But $\underset{\sim}{d}_2 = \underset{\sim}{d}_3 \wedge \underset{\sim}{d}_1 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \frac{-a}{\sqrt{\cdot}} \sin() & \frac{a}{\sqrt{\cdot}} \cos() & \frac{b}{\sqrt{\cdot}} \\ -\cos() & -\sin() & 0 \end{vmatrix} = \begin{pmatrix} \frac{b}{\sqrt{\cdot}} \sin(\ ) \\ -\frac{b}{\sqrt{\cdot}} \cos(\cdot) \\ \frac{a}{\sqrt{\cdot}} \end{pmatrix}$

$\therefore \boxed{\hat{\tau} = \underset{\sim}{d}_1' \cdot \underset{\sim}{d}_2 = \frac{b}{a^2+b^2}}$ [3]

(b) (ii)

$$\underset{\sim}{u} = (0, \kappa, \tau) \qquad \Rightarrow \quad \underset{\sim}{m} = K_2(\kappa - \hat{\kappa})\underset{\sim}{d}_2 + K_3(\tau - \hat{\tau})\underset{\sim}{d}_3$$

$$\hat{\underset{\sim}{u}} = (0, \hat{\kappa}, \hat{\tau})$$

$$\underset{\sim}{m}(1) = M\underset{\sim}{e}_z \Rightarrow K_2(\kappa - \hat{\kappa})\underset{\sim}{d}_2 \cdot \underset{\sim}{e}_z + K_3(\tau - \hat{\tau})\underset{\sim}{d}_3 \cdot \underset{\sim}{e}_z = M$$

Using, $\underset{\sim}{d}_2$, $\underset{\sim}{d}_3$ as computed in (iii), [3]

we get $$\boxed{\frac{K_2(\kappa - \hat{\kappa})a}{\sqrt{a^2+b^2}} + \frac{K_3(\tau - \hat{\tau})b}{\sqrt{a^2+b^2}} = M}$$

Noting $\kappa^2 + \tau^2 = \frac{1}{a^2+b^2} \Rightarrow a = \frac{\kappa}{\kappa^2+\tau^2}$, $b = \frac{\tau}{\kappa^2+\tau^2}$

$\therefore$ this can be written $$\underbrace{\frac{K_2(\kappa - \hat{\kappa})\kappa}{\sqrt{\kappa^2+\tau^2}} + \frac{K_3(\tau - \hat{\tau})\tau}{\sqrt{\kappa^2+\tau^2}}}_{\substack{\parallel \\ F_2(\kappa, \tau; \hat{\kappa}, \hat{\tau})}} = M \quad [1]$$

For the other relation, turn to force and moment balance: FB $\underset{\sim}{n}' = \underset{\sim}{0}$ w/ $\underset{\sim}{n}(1) = N\underset{\sim}{e}_z$

$\Rightarrow \underset{\sim}{n} = N\underset{\sim}{e}_z$ is constant $\leftarrow$ [1]

MB: $\underset{\sim}{m}' + \underset{\sim}{r}' \wedge \underset{\sim}{n} = \underset{\sim}{0}$ We have $\underset{\sim}{m}' = K_2(\kappa - \hat{\kappa})\underset{\sim}{d}_2' + K_3(\tau - \hat{\tau})\underset{\sim}{d}_3'$

$$= \left[-K_2\tau(\kappa - \kappa) + K_2\kappa(\tau - \hat{\tau})\right]\underset{\sim}{d}_1$$

and $$\underset{\sim}{r}' \wedge \underset{\sim}{n} = \underset{\sim}{d}_3 \wedge \underset{\sim}{n} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{j} \\ \frac{-a}{\sqrt{\cdot}}\sin(\ ) & \frac{a}{\sqrt{}}\cos(\ ) & \frac{b}{\sqrt{}} \\ 0 & 0 & N \end{vmatrix} = \frac{Na}{\sqrt{a^2+b^2}} \cdot \underbrace{\begin{pmatrix} \cos(\ ) \\ \sin(\ ) \\ 0 \end{pmatrix}}_{} = -\underset{\sim}{d}_1 \quad [1]$$

$\therefore$ we have $-K_2\tau(\kappa-\kappa)+K_2\kappa(\tau-\hat{\tau}) - \dfrac{Na}{\sqrt{a^2+b^2}} = 0$

$$\Rightarrow \underbrace{\left[-K_2\tau(\kappa-\hat{\kappa}) + K_2\kappa(\tau-\hat{\tau})\right]\cdot\frac{\sqrt{\kappa^2+\tau^2}}{\kappa}}_{\substack{\parallel \\ F_1(\kappa,\tau;\hat{\kappa},\hat{\tau})}} = N$$

[2]

② (a)

(i) Energy $\mathcal{E} = \int_{\Sigma} dS \left(\gamma + 2\kappa H^2 + \rho \hat{g} h \right)$

$= \int_U dx\,dy \sqrt{\det G} \left(\gamma + 2\kappa H^2 + \rho g h \right)$ Now use $\sqrt{\det G} = (1 + h_x^2 + h_y^2)^{\frac{1}{2}}$

$\sim 1 + \frac{1}{2}(h_x^2 + h_y^2) + O(|\nabla h^4|)$,

$$H = \frac{1}{2 g^{3/2}} \left(h_{xx}(1+h_y^2) + h_{yy}(1+h_x^2) - 2h_{xy} h_x h_y \right) \sim \frac{1}{2}\left(h_{xx} + h_{yy}\right) + O(|\nabla h^2|)$$

$\Rightarrow 2\kappa H^2 \sim \frac{1}{2}\kappa(\nabla^2 h)^2 + O(|\nabla h^2|)$ [2]

Thus, $\mathcal{E} \sim \int_U dx\,dy \left(\gamma + \frac{1}{2}\gamma|\nabla h|^2 + \frac{1}{2}\kappa(\nabla^2 h)^2 + \rho \hat{g} h + \text{h.o.t} \right)$

To minimize $\mathcal{E}$, set $\left.\frac{d}{d\varepsilon} \mathcal{E}[h + \varepsilon \eta(x,y)]\right|_{\varepsilon = 0} = 0$

$$\rightarrow 0 = \int_U dx\,dy \left(\gamma \underset{\substack{\parallel \\ \nabla\cdot(\nabla h \eta) - \eta\nabla^2 h}}{\nabla h \nabla \eta} + \kappa \underbrace{\nabla^2 h \nabla^2 \eta}_{= \nabla\cdot(\nabla\eta\nabla^2 h) - \nabla\nabla^2 h \nabla\eta} + \rho \hat{g} \eta \right)$$ [2]

$$= \int_{\partial U} ds \left(\underset{\substack{\uparrow \\ \gamma}}{\eta \nabla h} + \underset{\substack{\uparrow \\ \kappa}}{\nabla^2 h \nabla \eta} \right) \cdot \underset{\substack{\uparrow \\ \text{Normal vec} \\ \text{to } \partial U}}{\underset{\sim}{N}} - \int_U dx\,dy \left(\underset{\substack{\uparrow \\ \gamma}}{\nabla^2 h \eta} + \underset{\substack{\uparrow \\ \kappa}}{\,} \underbrace{\nabla\nabla^2 h \nabla \eta}_{= \nabla\cdot(\eta\nabla\nabla^2 h) - \eta\nabla^4 h} - \rho \hat{g} \eta \right)$$

$$= \int_{\partial U} ds \underbrace{\left(\eta [\gamma \nabla h - \kappa \nabla\nabla^2 h] + \nabla\eta \nabla^2 h \right) \cdot \underset{\sim}{N}}_{A} - \int_U dx\,dy \left(\gamma\nabla^2 h - \kappa\nabla^4 h - \rho \hat{g} \right) \eta$$

$\therefore$ h should satisfy $\kappa\nabla^4 h - \gamma\nabla^2 h = -\rho\hat{g} \Rightarrow \boxed{\nabla^4 h - \frac{1}{\lambda^2}\nabla^2 h = -\mu}$

with $\boxed{\lambda := \sqrt{\frac{\kappa}{\gamma}}, \quad \mu := \frac{\rho \hat{g}}{\kappa}}$

ii) BC if $h + \alpha \overset{\substack{\nabla h \cdot N \\ \parallel}}{\frac{\partial h}{\partial N}} = r(x,y)$ on ∂U, we must require   [3]

$\eta + \alpha \frac{\partial \eta}{\partial N} = 0$ on ∂U. The bdy term A becomes

$(\gamma \nabla h - \kappa \nabla\nabla^2 h)\cdot \underset{\sim}{N} - \frac{1}{\alpha}\nabla^2 h$. setting this to zero forms the 2nd boundary condition.

(iii) We are given $\gamma = 10^{-3}\ N/mm$, $\kappa = 10^{-3}\ N\,mm$, $\rho = \frac{1\,g}{cm^2}$, $\hat{g} = \frac{10m}{s^2}$

- we scale lengths by L : $h' = \frac{h}{L}$, $x' = \frac{x}{L}$, $y' = \frac{y}{L}$

$$\rightarrow \underbrace{\frac{\kappa \nabla'^4 h'}{L^3}}_{\text{Bending}} - \underbrace{\frac{\gamma \nabla'^2 h'}{L}}_{\text{Stretching}} = \underbrace{-\rho \hat{g}}_{\text{Gravity}} \Rightarrow \nabla'^4 h' - \alpha \nabla'^2 h' = -\beta$$

$$\alpha = \frac{L^2 \gamma}{\kappa} = \frac{L^2}{mm^2}$$

[2]

$$\beta = \frac{L^3 \rho \hat{g}}{\kappa} = L^3 \frac{1\,\cancel{g}}{\cancel{cm^2}} \cdot \frac{10\,m}{\cancel{s^2}} \cdot \frac{10^3}{\underset{\substack{\shortparallel \\ \frac{kg\,m}{\cancel{s^2}}}}{N}\,mm} \cdot \frac{10^{-3}\,\cancel{kg}}{\cancel{g}} \cdot \frac{10^{-2}\,\cancel{cm^2}}{mm^2} = 10^{-1} \frac{L^3}{mm^3}$$

$\therefore$ Stretching ~ Gravity if $\alpha \sim \beta$, ie $L \sim 10\,mm$

- For $L \sim 10\,mm$, $\alpha \sim \beta \sim 100$, so bending negligible
- For bending & surface tension to balance, need $\alpha \sim 1 \rightarrow L \sim 1\,mm$

  $\rightarrow \beta \sim 10^{-1}$ so gravity negligible [3]

(b) (i) $h = h(x)$ only w/ $\gamma = \gamma(x)$

The energy is $\mathcal{E}[h(x)] = \int_{U_x} dx\,\left(\gamma + \frac{1}{2}\gamma h'(x)^2 + \frac{1}{2}\kappa h''(x)^2 + \rho\hat{g} h\right)$

Then $\frac{d}{d\varepsilon}\mathcal{E}[h+\varepsilon\eta(x)]\Big|_{\varepsilon=0} = \int_{U_x} dx\,\left(\gamma(x)h'(x)\eta'(x) + \kappa h''(x)\eta''(x) + \rho\hat{g}\eta\right)$

$$= \gamma h'\eta + \kappa h''\eta' - \kappa h'''\eta\Big|_{\partial U_x} - \int_{U_x} dx\left[\left(\frac{d}{dx}(\gamma h') - \kappa h'''' - \rho\hat{g}\right)\eta(x)\right]$$

int. by parts twice

∴ The Euler-Lagrange eq'n for h is

$$\boxed{\kappa h''''(x) - \frac{d}{dx}\left(\gamma(x)h'(x)\right) = -\rho\hat{g}}$$

[4]

(ii) We have $\kappa = 0$, $\gamma = \gamma_0\left(1 - \varepsilon e^{-x^2/2\sigma^2}\right)$

$\rightarrow \frac{d}{dx}\left(\gamma(x)h'(x)\right) = \rho\hat{g}$, w/ $h(\pm\frac{L}{2}) = 0$

We seek $h \sim h_0 + \varepsilon h_1 + \ldots$, and write $\gamma = \gamma_0 - \varepsilon\gamma_p$

$\rightarrow \frac{d}{dx}\left((\gamma_0 - \varepsilon\gamma_p)(h_0' + \varepsilon h_1' \ldots)\right) = \rho\hat{g}$ $\quad\left(\gamma_p := \gamma_0\exp\left(\frac{-x^2}{2\sigma^2}\right)\right)$

$\Rightarrow \frac{d}{dx}\left(\gamma_0 h_0' + \varepsilon(\gamma_0 h_1' - \gamma_p h_0') \ldots\right) = \rho\hat{g}$

$\underline{O(1)}$ $\quad \underline{h_0'' = \frac{\rho\hat{g}}{\gamma_0}}$ w/ $h_0(\pm\frac{L}{2}) = 0$

[3]

has sol'n $\boxed{h_0(x) = \frac{\rho\hat{g}}{2\gamma_0}\left(x^2 - \frac{L^2}{4}\right)}$

so sag of healthy tissue is $h_0(0) = -\frac{\rho\hat{g}L^2}{8\gamma_0}$

At $O(\varepsilon)$, $\frac{d}{dx}\left(\gamma_0 h_1' - \gamma_p h_0'\right) = 0 \Rightarrow h_1'' = \frac{d}{dx}\left(e^{-x^2/2\sigma^2} \cdot h_0'\right)$

$$\Rightarrow h_1' = c_1 + e^{-x^2/2\sigma^2} \cdot \frac{\rho g x}{\gamma_0}$$

[1]

$$\Rightarrow h_1(x) = c_1 x + \frac{\rho \hat{g}}{\gamma_0} \underbrace{\int x e^{-x^2/2\sigma^2} dx}_{= -\sigma^2 e^{-x^2/2\sigma^2}} + c_2$$

$\left(u = e^{-x^2/2\sigma^2}, \quad du = -\frac{2x}{2\sigma^2} dx\right)$

We have

$$h_1 = -\frac{\rho \hat{g} \sigma^2}{\gamma_0} e^{-x^2/2\sigma^2} + c_1 x + c_2$$

. Apply BC:

$$0 = -\frac{\rho \hat{g} \sigma^2}{\gamma_0} e^{-L^2/8\sigma^2} \pm \frac{c_1 L}{2} + c_2 \Rightarrow$$

$c_1 = 0$,

$c_2 = \frac{\rho \hat{g} \sigma^2}{\gamma_0} \exp\left(\frac{-L^2}{8\sigma^2}\right)$

[3]

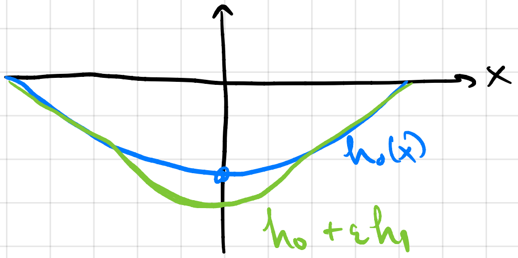
Giving the soln

$$\boxed{h_1(x) = \frac{\rho \hat{g} \sigma^2}{\gamma_0}\left(e^{-L^2/8\sigma^2} - e^{-x^2/2\sigma^2}\right)}$$

∴ The change in sag due to tissue damage is

$$\varepsilon h_1(0) = \varepsilon \frac{\rho \hat{g} \sigma^2}{\gamma_0}\left(e^{-L^2/8\sigma^2} - 1\right)$$

$h_1(0) < 0 \Rightarrow$ - The tissue sags more



[2]

③ (a)

(i) Volumetric growth satisfies $dv = \eta \, dV$

current volume element

ref. vol. element

$dV = R^2 \sin\varphi \, d\varphi \, d\theta \, dR$

$dv = r^2 \sin\varphi \, d\varphi \, d\theta \, dr$

$\Rightarrow r^2 dr = \eta R^2 dR$

so $\frac{\partial r}{\partial R} = \eta \left(\frac{R}{r}\right)^2$ [1]

(ii) Diffusion eqn is $u_t = D\nabla^2 u - Q$

· we can assume quasistatic equil, $u_t \approx 0$, if the diffusion timescale is much shorter than the growth timescale, defined in $\dot{\eta} = k\eta u$

· setting $u_t = 0 \rightarrow D\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = Q$

$\Rightarrow r^2 \frac{du}{dr} = \frac{Q}{3D} r^3 + c_1$ Now apply $\frac{\partial u}{\partial r} = 0$ at $r = a = A$

$\rightarrow c_1 = -\frac{Qa^3}{3D}$. Then $\frac{du}{dr} = \frac{Q}{3D}\left(r - \frac{a^3}{r^2}\right)$

$\Rightarrow u = \frac{Q}{3D}\left(\frac{r^2}{2} + \frac{a^3}{r}\right) + c_2$, & $u(b) = U$ [2]

$c_2 = U - \frac{Q}{3D}\left(\frac{b^2}{2} + \frac{a^3}{b}\right)$

However, we require $u \geq 0$, so formula above will cease to be valid if b large enough that $u \searrow 0$ before $r = a$. [1]

- in which case, we replace no flux cond. w/ $u(\hat{a}) = 0$, w/ $\hat{a}$ det'd via $\frac{\partial u}{\partial r}(\hat{a}) = 0$

∘ we switch to this scenario when $u(a)=0$ in previous sol'n. ∴ b_{crit} det'd from

$$u(a) = \frac{Q}{3D}\left(\frac{a^2}{2}+\frac{a^3}{a}\right) + U - \frac{Q}{3D}\left(\frac{b^2}{2}+\frac{a^3}{b}\right) = 0$$

[2]

ie $$\frac{Q}{6D}b^3 + \left(\frac{Q}{2D}a^2 + U\right)b - \frac{Q}{3D}a^3 = 0$$

(iii) $$\int_a^b r^2\,dr = \int_A^B \eta R^2\,dR \Rightarrow \tfrac{1}{3}(b^3-a^3) = \int_A^B \eta R^2\,dR$$

$$\xRightarrow{\frac{\partial}{\partial t}} \quad b^2\frac{db}{dt} = \int_A^B \underset{\overset{\shortparallel}{ku\eta}}{\dot\eta}\, R^2\,dR = \int_A^B ku\,\underset{\overset{\shortparallel}{r^2dr}}{\underline{\eta R^2 dR}} = \int_a^b k u r^2\,dr$$

[3]

Now plug in $$u(r) = \frac{Q}{3D}\left(\frac{r^2}{2}+\frac{a^3}{r}\right) + U - \frac{Q}{3D}\left(\frac{b^2}{2}+\frac{a^3}{b}\right) = 0$$

Then $$\int_a^b ur^2\,dr = \frac{Q}{3D}\left(\frac{b^5-a^5}{10} + \frac{a^3}{2}(b^2-a^2)\right) + \left[U - \frac{Q}{3D}\left(\frac{b^2}{2}+\frac{a^3}{b}\right)\right]\cdot\frac{b^3-a^3}{3} \quad \bigstar$$

For a, b both $O(\varepsilon)$, $\bigstar \sim \frac{U}{3}(b^3-a^3) + O(b^5)$

so $$\frac{db}{dt} \sim \frac{kU}{3}\left(\frac{b^3-a^3}{b^2}\right) \to \frac{b^2}{b^3-a^3}\,db = \frac{kU}{3}\,dt$$

[2]

$$\tfrac{1}{3}\ln(b^3-a^3) = \frac{kUt}{3} + c$$

$$b(0)=B \Rightarrow c = \tfrac{1}{3}\ln(B^3- \ ^3) \Rightarrow b^3 = a^3 + (B^3-A^3)e^{kUt}$$

so early behaviour: $$b \sim \left\{a^3 + (B^3-A^3)e^{kUt}\right\}^{\frac{1}{3}}$$

[1]

③ (b)

(i)

$$F = \mathrm{diag}\left(r'(R), \frac{r}{R}, \frac{r}{R}\right), \quad A = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_2),$$

$$G = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_2)$$

$$F = AG \Rightarrow r'(R) = \gamma_1 \alpha_1, \quad \frac{r}{R} = \alpha_2 \gamma_2$$

[3]

• incompressible $\Rightarrow \alpha_1 \alpha_2^2 = 1 \Rightarrow \dfrac{r'(R)}{\gamma_1} \cdot \dfrac{r^2}{\gamma_2^2 R^2} = 1$

$$\Rightarrow r^2 dr = \gamma_1 \gamma_2^2 R^2 dR \Rightarrow \boxed{r^3 - A^3 = \gamma_1 \gamma_2^2 (R^3 - A^3)}$$

$(\&\ r(A) = A)$

(ii) We are given $W(\alpha_1, \alpha_2, \alpha_2)$ and

$\mathrm{div}\, T = 0$ as $\dfrac{dt_1}{dr} + 2\dfrac{t_1 - t_2}{r} = 0$

The constitutive law $T = A\dfrac{\partial W}{\partial A} - p\mathbb{1}$ reads

in component form

$$\begin{cases} t_1 = \alpha_1 \dfrac{\partial W}{\partial \alpha_1} - p \\ t_2 = \alpha_2 \dfrac{\partial W}{\partial \alpha_2} - p \\ t_3 = \alpha_3 \dfrac{\partial W}{\partial \alpha_3} - p \end{cases}$$

← But $t_3 = t_2$, $\alpha_3 = \alpha_2$

If define $\alpha = \alpha_2$, then $\alpha_1 = \dfrac{1}{\alpha^2}$

and $\hat{W}(\alpha) = W(\alpha^{-2}, \alpha, \alpha)$

$\left[W_i := \dfrac{\partial W}{\partial \alpha_i} \right]$

$$\Rightarrow \hat{W}'(\alpha) = -\frac{2}{\alpha^3} W_1 + \underbrace{W_2 + W_3}_{\shortparallel\, 2W_2} \Rightarrow \alpha \hat{W}'(\alpha) = 2(\alpha W_2 - \alpha_1 W_1)$$

[2]

$$\Rightarrow \quad \alpha \hat{W}'(\alpha) = 2(t_2 - t_1) \Rightarrow \frac{\partial t_1}{\partial r} = \frac{\alpha \hat{W}'(\alpha)}{r}$$

Now, $\frac{d}{dr} = \frac{d}{d\alpha}\frac{d\alpha}{dr}$ & $\alpha = \frac{r}{\gamma_2 R} \Rightarrow \frac{d\alpha}{dr} = \frac{1}{\gamma_2 R} - \underbrace{\frac{r}{\gamma_2 R^2}}_{\frac{\alpha}{R}} \underbrace{\frac{dR}{dr}}_{\frac{\alpha^2}{\gamma_1}}$

$$\Rightarrow r\frac{d}{dr} = \frac{d}{d\alpha} \overset{\gamma_2 \alpha}{\boxed{\frac{r}{R}}} \left(\frac{1}{\gamma_2} - \frac{\alpha^3}{\gamma_1} \right)$$

[2]

$$\Rightarrow \frac{dt_1}{d\alpha} = \frac{\gamma_1 \hat{W}'(\alpha)}{\gamma_1 - \gamma_2 \alpha^3}$$

On outer edge, $\alpha = \frac{b}{\gamma_2 B} =: \alpha_B$,

On inner edge, $\alpha = \frac{1}{\gamma_2} =: \alpha_A$

$$\Rightarrow t_1(A) = -\int_{\alpha_A}^{\alpha_B} \frac{\partial t_1}{\partial \alpha} d\alpha = -\int_{\alpha_A}^{\alpha_B} \frac{\gamma_1 \hat{W}'(\alpha)}{\gamma_1 - \gamma_2 \alpha^3} d\alpha$$

$$W = \frac{\mu}{2}(\alpha_1^2 + 2\alpha_2^2 - 3) \Rightarrow \hat{W}(\alpha) = \frac{\mu}{2}(\alpha^{-4} + 2\alpha^2 - 3)$$

So $\hat{W}'(\alpha) = 2\mu(\alpha - \alpha^{-5})$, thus [2]

$$\boxed{t_1(A) = 2\mu\gamma_1 \int_{\alpha_A}^{\alpha_B} \frac{\alpha^{-5} - \alpha}{\gamma_1 - \gamma_2 \alpha^3} d\alpha}$$

(iii)

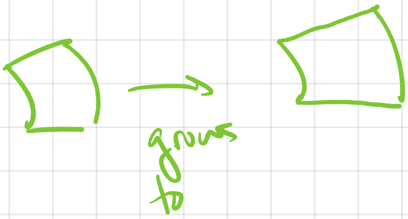
(B) is the case $\gamma_1 = 1$, $\gamma_2 = 1.2$

- circumferential growth plus adhesion to fixed core creates compressive hoop stress throughout. [2]

- and material is stretched in radial direction since no radial growth $\Rightarrow t_1 \geq 0$

(A) is the case $\gamma_1 = 1.5$, $\gamma_2 = 1$

- radial growth generates radial compression if no circumferential growth. [2]



- outer surface is stretched in circumferential direction, so hoop stress $t_2(B) > 0$

$$\alpha = \frac{r}{\gamma_2 R} \quad , \quad \alpha_1 = \frac{r'(R)}{\gamma_1} = \frac{1}{\alpha^2} \Rightarrow \frac{dr}{dR} = \frac{\gamma_1}{\alpha^2}$$

$$\frac{dr}{d\alpha} = \frac{dr}{dR}\frac{dR}{d\alpha} \qquad r = \gamma_2 \alpha R \rightarrow \frac{dr}{dR} = \gamma_2 \alpha + \gamma_2 R \frac{d\alpha}{dR} = \frac{\gamma_1}{\alpha^2}$$

$$\frac{d\alpha}{dR} = \frac{r'(R)}{\gamma_2 R} - \frac{r}{\gamma_2 R^2} = \underbrace{\frac{\gamma_1}{\gamma_2 \alpha^2 R} - \frac{\alpha}{R}}_{\frac{1}{R}\left(\frac{\gamma_1 - \gamma_2 \alpha^3}{\gamma_2 \alpha^2}\right)} = \frac{d\alpha}{dr} \cdot \frac{\gamma_1}{\alpha^2}$$

$$\Rightarrow \quad \alpha \hat{W}'(\alpha) = 2(t_2 - t_1) \Rightarrow \frac{\partial t_1}{\partial r} = \frac{\alpha \hat{W}'(\alpha)}{r}$$

(iii) Outer edge is stress free $\Rightarrow$ $t_1(b) = 0$

Thus, $t_1(A) = -\int_A^b \frac{\partial t_1}{\partial r} dr = -\int_A^b \frac{\alpha \hat{W}'(\alpha)}{r} dr$

- Now, $W = \frac{\mu}{2}(\alpha_1^2 + 2\alpha_2^2 - 3) \Rightarrow \hat{W} = \frac{\mu}{2}(\alpha^{-4} + 2\alpha^2 - 3)$

$$\Rightarrow \alpha \hat{W}'(\alpha) = -2\mu(\alpha^{-4} - 2\alpha^2)$$

and $\alpha = \frac{r}{\gamma_2 R} = \frac{(A^3 + \gamma_1\gamma_2^2(B^3 - A^3))^{\frac{1}{3}}}{\gamma_2 R} = \alpha(R)$

and $dr = \alpha_1 \gamma_1 dR = \frac{\gamma_1}{\alpha^2} dR$

$$\Rightarrow \boxed{t_1(A) = 2\mu \int_A^b \frac{\alpha(R)^{-4} - 2\alpha(R)^2}{r(R)} \cdot \gamma_1 \, dR}$$