PROJECTIVE GEOMETRY – SHEET 1

Projective Spaces. Homogeneous Co-ordinates. Projective Transformations. General Position. Cross-ratio.
(Exercises on lectures 1–4)

In these questions, $\mathbb{F}$ denotes the base field.

1.(i) If we identify $(x, y) \in \mathbb{F}^2$ with the point $[1\colon x\colon y] \in \mathbb{FP}^2$, what is the point at infinity shared by all lines of the form $y = mx + c$, where m is fixed?
(ii) Show that those projective transformations in $PGL(3, \mathbb{F})$ which map the line at infinity to itself form a subgroup of $PGL(3, \mathbb{F})$ which is isomorphic to

$$AGL(2, \mathbb{F}) = \left\{ \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b} : A \in GL(2, \mathbb{F}), \mathbf{b} \in \mathbb{F}^2 \right\}.$$

Which of these mappings fix the line at infinity pointwise?

2.(i) Let $\mathbb{P}(U_1)$ and $\mathbb{P}(U_2)$ be two non-intersecting lines in the 3-dimensional projective space $\mathbb{FP}^3 = \mathbb{P}(\mathbb{F}^4)$. Show that

$$\mathbb{F}^4 = U_1 \oplus U_2.$$

(ii) Deduce that three pairwise non-intersecting lines in $\mathbb{FP}^3$ have infinitely many transversals, i.e. projective lines meeting all three.

3. Let L_1, L_2 be two (nonempty) projective linear subspaces of a projective space $\mathbb{P}(V)$, corresponding to linear subspaces $U_1, U_2 \subset V$. Show that the span

$$\langle L_1, L_2 \rangle = \mathbb{P}(U_1 + U_2)$$

is the union of projective lines P_1P_2 with $P_i \in L_i$.

4.(i) List the elements of $PGL(2, \mathbb{F}_2)$. What is the order of $PGL(2, \mathbb{F})$ if $|\mathbb{F}| = q$?
(ii) By considering the action of $PGL(2, \mathbb{F}_2)$ on $\mathbb{F}_2\mathbb{P}^1$, show that $PGL(2, \mathbb{F}_2) \cong S_3$. Is $PGL(2, \mathbb{F}_3) \cong S_4$? Is $PGL(2, \mathbb{F}_5) \cong S_6$?

5. Let a, b, c, d be four distinct points in $\mathbb{C}$. Show that a, b, c, d lie on a circline if and only if the cross-ratio $(ab : cd)$ is real.

6. We say x_0, x_1 and x_2, x_3 are *harmonically separated* if $(x_0x_1 : x_2x_3) = -1$, where the x_i are distinct points in a projective line $\mathbb{FP}^1$. Let a, b, c, d be four points in general position in the projective plane $\mathbb{FP}^2$ and let e, f, g be the diagonal points, i.e. $e = ac \cap bd, f = ab \cap cd, g = ad \cap bc$. Let ge meet ab at h. Prove that a, b and h, f are harmonically separated.

7. (i) Let $\tau \in PGL(2, \mathbb{C})$, other than the identity. Show that τ fixes either one or two points. Show that this need not be true over other fields.
(ii) If τ fixes two points, show that there is an inhomogeneous co-ordinate z on $\mathbb{CP}^1$ with respect to which

$$\tau(z) = \lambda z, \qquad \lambda \neq 0, 1.$$

Is the same true over other fields?
(iii) Let A_1, A_2, A_3 be three distinct points in $\mathbb{CP}^1$ and let $n \geqslant 3$ be an integer. Show that there is $\tau \in PGL(2, \mathbb{C})$ such that $\tau(A_1) = A_2, \tau(A_2) = A_3$ and τ has order n.

8. Use the strategy outlined in the lectures to prove Pappus's Theorem: *Let A, B, C and A', B', C' be two collinear triples of distinct points in the projective plane $\mathbb{FP}^2$. Then the three intersection points $AB' \cap A'B, BC' \cap B'C$ and $CA' \cap C'A$ are collinear.* Proceed by the following steps.
(i) Prove the theorem in the degenerate case when A, B, C', B' are *not* in general position.
(ii) If these 4 points are in general position, explain why without loss of generality we may take them to be

$$A = [1, 0, 0], \quad B = [0, 1, 0], \quad C' = [0, 0, 1], \quad B' = [1, 1, 1].$$

9. (Optional) Every line in the real affine plane $\mathbb{R}^2$ can be written in the form $ax + by + c = 0$ where a, b are not both zero. Of course, $\lambda ax + \lambda by + \lambda c = 0$ is an equation of the same line where $\lambda \neq 0$. Hence the space of lines can be identified with

$$M = \frac{\mathbb{R}^2 \backslash \{(0, 0)\} \times \mathbb{R}}{\mathbb{R}^*}.$$

Identify M as a subspace of $\mathbb{RP}^2$. What is the topology of M?