2021 Question 1

a) Modelling the Earth as a spherically symmetric and non-rotating body in an otherwise empty spacetime, explain why the metric outside of the Earth is given by the Schwarzschild metric.
Solution: Birkhoff's theorem tells us that the metric outside of a spherically symmetric body must be given by the Schwarzschild metric.

b) A satellite orbits the Earth in a circular orbit at a radius r_S (in Schwarzschild coordinates), and Alice stands on the surface of the Earth, which has radius r_A. No external forces act on the satellite. Each time the satellite passes directly above Alice, it emits a radial light ray, which is then received by Alice.
Let the proper time along the worldline of the satellite be τ_S, and let the proper time along Alice's worldline be τ_A. At the moment when the satellite first emits a light ray, its clock reads $\tau_S = 0$, and when Alice first receives the light ray, her clock reads $\tau_A = 0$. Subsequent signals are emitted by the satellite when its clock reads $\tau_S = T_{(S,n)}$ (for $n \in \mathbb{N}$), and received by Alice when her clock reads $\tau_A = T_{(A,n)}$.

(i) For fixed $n \geq 1$, find $T_{(A,n)}$ as a function of $T_{(S,n)}$, r_S and r_A, justifying all of your calculations.

Solution:
We will calculate τ_S and τ_A as functions of the Schwarzschild time coordinate t. Then, let the Schwarzschild time for a radial light ray to go from r_S to r_A be t_0 (in the end we won't have to calculate this constant).
First we calculate $\tau_S(t)$. The worldline of the satellite, written in Schwarzschild coordinates, is $(t(\tau_S), r(\tau_S), \theta(\tau_S), \phi(\tau_S))$. This worldline extremises the Lagrangian

$$L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2,$$

where 'dots' are derivatives with respect to τ_S.

First we note that, without loss of generality, we can take the worldline of the satellite to lie in the equatorial plane $\theta = 0$. This is because the Euler-Lagrange equation for θ is

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2 \sin\theta \cos\theta \dot{\phi} = 0,$$

and $\theta \equiv 0$ is a solution to this ODE. Hence, if the initial conditions are $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, then this is the (unique) solution, and these initial conditions can be obtained by using the isometries associated with spherical symmetry.

Next we note that, since the Lagrangian is independent of t, E is constant, where

$$E = -\frac{1}{2}\frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{2M}{r}\right)\dot{t}.$$

Similarly, since the Lagrangian is independent of ϕ, Ω is constant, where

$$\Omega = \frac{1}{2}\frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2\theta \dot{\phi}.$$

Finally, since τ_S is the proper time along the worldline of the satellite, $L = -1$.

Putting this together, we find that

$$-1 = -\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^{-2}\Omega^2$$

$$\Rightarrow \frac{1}{2}\dot{r}^2 - \frac{M}{r} + \frac{\Omega^2}{2r^2} - \frac{M\Omega^2}{r^3} = \frac{1}{2}(E^2 - 1),$$

so the motion corresponds to the motion of a particle in one dimension, with potential $V(r) = -\frac{M}{r} + \frac{\Omega^2}{2r^2} - \frac{M\Omega^2}{r^3}$ and energy $\frac{1}{2}(E^2 - 1)$.

For a circular orbit we must have $\dot{r} = \ddot{r} = 0$, meaning that we must be at a local extremum of the potential energy (since $\ddot{r} = -V'(r)$). Since the satellite orbits at a radius r_S, the extrema of V are at r_S, where

$$Mr_S^2 - \Omega^2 r_S + 3M\Omega^2 = 0.$$

Hence the angular momentum of the satellite is given by

$$\Omega^2 = \frac{Mr_S^2}{r_S - 3M}.$$

Next, since $\dot{r} = 0$ the energy of the orbit is given by

$$E^2 = \left(1 - \frac{2M}{r_S}\right)\left(1 + r_S^{-2}\Omega^2\right) = \frac{(r_S - 2M)^2}{r_S(r_S - 3M)}.$$

Finally, recall that $\dot{t} = \left(1 - \frac{2M}{r_S}\right)^{-1} E > 0$. Hence

$$\frac{\mathrm{d}t}{\mathrm{d}\tau_S} = \sqrt{\frac{r_S}{r_S - 3M}}.$$

Hence we have

$$\tau_S = C_S + \sqrt{\frac{r_S - 3M}{r_S}}\, t,$$

for some constant C_S.

Next we compute the proper time along the worldline of Alice. Along Alice's worldline, all of the spatial coordinates are constant, and so we have

$$-1 = g_{ab} \frac{\mathrm{d}x^a}{\mathrm{d}\tau_A} \frac{\mathrm{d}x^b}{\mathrm{d}\tau_A} = -\left(1 - \frac{2M}{r_A}\right) \left(\frac{\mathrm{d}t}{\mathrm{d}\tau_A}\right)^2 .$$

Hence we have

$$\tau_A = C_A + \sqrt{\frac{r_A - 2M}{r_A}}\, t,$$

for some constant C_A.

Now, since the coordinate time for a radial light ray to reach Alice from Bob is t_0, which is a constant independent of n, we find that

$$T_{(A,n)} = \sqrt{\frac{r_S(r_A - 2M)}{r_A(r_S - 3M)}}\, T_{(S,n)}.$$

(ii) Hence show that $r_S = \frac{3}{2} r_A$ is the unique radius at which it is possible for the clock on the satellite and Alice's clock to be synchronised, so that $T_{(A,n)} = T_{(S,n)}$ for all n.

Solution:

If the two clocks are synchronised, then we must have

$$\frac{r_S(r_A - 2M)}{r_A(r_S - 3M)} = 1$$

$$\Rightarrow r_S = \frac{3}{2} r_A.$$

c) Bob claims that, because the satellite is moving fast relative to Alice, time dilation will cause the clock on the satellite to run slower than Alice's clock, regardless of the altitude of the satellite. Referring to your answer to part (**??**), explain why Bob is correct or incorrect. If he is incorrect, what physical effect is Bob failing to take into account?

Bob is not correct: for low altitudes ($r_A \sim r_S$), we have

$$T_{(A,n)} \sim \sqrt{\frac{(r_S - 2M)}{(r_S - 3M)}}\, T_{(S,n)},$$

and so $T_{(A,n)} > T_{(S,n)}$, and Bob appears to be correct. However, for very large altitudes, $r_S \to \infty$, we see that

$$T_{(A,n)} \sim \sqrt{1 - \frac{2M}{r_A}}\, T_{(S,n)},$$

and so $T_{(A,n)} < T_{(S,n)}$, i.e. "Alice's clock runs slower than the clock on the satellite". The physical effect that causes this is the gravitational time dilation.

2017 Question 1 part (c)

Consider

$$s_{abcd}(x) = \epsilon_{abcd}\omega(x),$$

where we define ϵ_{abcd} by setting $\epsilon_{0123} = 1$, as well as $\epsilon_{[abcd]} = \epsilon_{abcd}$ in every coordinate system. Given that s_{abcd} transforms as a $(0,4)$-tensor under (1), derive the transformation behaviour of ω under (1). Find how the covariant derivative ∇_a must act on ω.

[(1) is coordinate transformations]

Unfortunately this question doesn't give you enough information to solve it. Under the map

$$\nabla_a\omega \mapsto \nabla_a\omega + X_a$$
$$\nabla_a\epsilon_{bcde} \mapsto \nabla_a\epsilon_{bcde} - X_a\epsilon_{bcde}$$

the expression $\nabla_a(\omega\epsilon_{abcd})$ is invariant. Since we don't know how the covariant derivative acts on ϵ_{abcd} (it's not a tensor!), given what we're told in the question, we can't uniquely determine the expression $\nabla_a\omega$, since there is always this ambiguity.
The missing information is this: the covariant derivative acts on ϵ as

$$\nabla\epsilon = 0.$$

Using the normal expressions for tensors:

$$\begin{aligned}\nabla_a s_{bcde} &= \partial_a s_{bcde} - \Gamma^f_{\ ab} s_{fcde} - \Gamma^f_{\ ac} s_{afde} - \Gamma^f_{\ ad} s_{bcfe} - \Gamma^f_{\ ae} s_{bcdf} \\ &= (\partial_a \omega)\epsilon_{bcde} - \omega\left(\Gamma^f_{\ ab}\epsilon_{fcde} + \Gamma^f_{\ ac}\epsilon_{afde} + \Gamma^f_{\ ad}\epsilon_{bcfe} + \Gamma^f_{\ ae}\epsilon_{bcdf}\right)\end{aligned}$$

On the other hand we have $\nabla_a s_{bcde} = (\nabla_a \omega)\epsilon_{bcde}$.
Setting $(bcde) = (0123)$ relative to some arbitrary coordinate system, we find

$$\begin{aligned}\nabla_a \omega &= \partial_a \omega - \omega\left(\Gamma^f_{\ a0}\epsilon_{f123} + \Gamma^f_{\ a1}\epsilon_{0f23} + \Gamma^f_{\ a2}\epsilon_{01f3} + \Gamma^f_{\ a3}\epsilon_{012f}\right) \\ &= \partial_a \omega - \omega\left(\Gamma^0_{\ a0}\epsilon_{0123} + \Gamma^1_{\ a1}\epsilon_{0123} + \Gamma^2_{\ a2}\epsilon_{0123} + \Gamma^3_{\ a3}\epsilon_{0123}\right) \\ &= \partial_a \omega - \omega\left(\Gamma^0_{\ a0} + \Gamma^1_{\ a1} + \Gamma^2_{\ a2} + \Gamma^3_{\ a3}\right) \\ &= \partial_a \omega - \Gamma^b_{\ ab}\omega\end{aligned}$$

Can also check that this transforms like a vector times ω.

2018 Question 2

Consider a metric of the form

$$\mathrm{d}s^2 = -e^{-2\Lambda(r)}\mathrm{d}t^2 + e^{2\Lambda(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right). \tag{1}$$

For the Schwarzschild solution, where

$$e^{-2\Lambda(r)} = 1 - \frac{2MG}{r}, \tag{2}$$

(c) The Einstein tensor of the space-time (1) has a component

$$G_{tt} = e^{-2\Lambda}\left(\frac{2e^{-2\Lambda}}{r}\frac{\partial\Lambda}{\partial r} + \frac{1-e^{-2\Lambda}}{r^2}\right). \tag{3}$$

Use this to derive the form of the Schwarzschild solution (2).

Vacuum Einstein equations are $G_{\mu\nu} = 0$, so

$$0 = \frac{2e^{-2\Lambda}}{r}\frac{\mathrm{d}\Lambda}{\mathrm{d}r} + \frac{1 - e^{-2\Lambda}}{r^2}$$

Now, let $\nu = e^{-2\Lambda}$, so $\mathrm{d}\nu = -2e^{-2\Lambda}\mathrm{d}\Lambda$. Hence

$$-\frac{1}{r}\frac{\mathrm{d}\nu}{\mathrm{d}r} + \frac{1-\nu}{r^2} = 0,$$

and so

$$\int \frac{\mathrm{d}r}{r} = \int \frac{\mathrm{d}\nu}{1-\nu}$$
$$\Rightarrow \log r + C = -\log(1-\nu)$$
$$\Rightarrow \nu = 1 - \frac{e^{-C}}{r}$$

Now choose the integration constant $C = -\log(2GM)$.

(d) Consider a spherically symmetric charge distribution $\rho(r)$ with $\rho(r) = 0$ for $r > r0$ in the space-time (1). Such a charge distribution induces a vector potential A, for which the only non-vanishing component is

$$A_t = Q/r.$$

(i) Find the associated field strength and the energy-momentum tensor
(ii) Using the expression (3) for G_{tt}, find the function $\Lambda(r)$ induced by such an energy momentum tensor

(i) Given a four-potential A, the associated field strength is $F = \mathrm{d}A$ or, with indices,

$$F_{ab} = \partial_a A_b - \partial_b A_a$$

(it doesn't matter whether we use ∂ or ∇ here, by the torsion-free property). Hence, if $A = (Q/r)\mathrm{d}t$, we have

$$F_{rt} = -F_{tr} = -\frac{Q}{r^2}$$

with all other components of the field strength vanishing.

The energy-momentum tensor in electrodynamics is

$$T_{ab} = F_a{}^c F_{bc} - \frac{1}{4}(F^{cd}F_{cd})g_{ab}.$$

We calculate

$$\begin{aligned} F^{cd}F_{cd} &= F^{tr}F_{tr} + F^{rt}F_{rt} = 2F^{tr}F_{tr} \\ &= 2(g^{-1})^{tt}(g^{-1})^{rr}(F_{tr})^2 = -\frac{2Q^2}{r^4} \end{aligned}$$

and so we find

$$T_{tt} = \frac{1}{2}\frac{Q^2}{r^4}e^{-2\Lambda}$$

$$T_{rr} = -\frac{1}{2}\frac{Q^2}{r^4}e^{2\Lambda}$$

$$T_{\theta\theta} = \frac{1}{2}\frac{Q^2}{r^4}r^2$$

$$T_{\phi\phi} = \frac{1}{2}\frac{Q^2}{r^4}r^2\sin^2\theta$$

with the off-diagonal components being zero.

To find the associated function Λ we use the tt component of the Einstein equations:

$$G_{tt} = 8\pi T_{tt}$$

which gives us

$$e^{-2\Lambda}\left(\frac{2e^{-2\Lambda}}{r}\frac{\partial\Lambda}{\partial r} + \frac{1-e^{-2\Lambda}}{r^2}\right) = 4\pi\frac{Q^2}{r^4}e^{-2\Lambda}$$

Following the same ideas as before:

$$\begin{aligned} \frac{\mathrm{d}\nu}{\mathrm{d}r} &= \frac{1-\nu}{r} - \frac{4\pi Q^2}{r^3} \\ \Rightarrow \int \frac{1}{1-\nu}\mathrm{d}\nu &= \int\left(\frac{1}{r} - \frac{4\pi Q^2}{r^3}\right). \end{aligned}$$

Now try setting $\nu = \nu_S + \nu_1$, where ν_S is the function ν that we found before. In particular,

$$\frac{\mathrm{d}\nu_S}{\mathrm{d}r} = \frac{1-\nu_S}{r}$$

and so we see that ν_1 must satisfy

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}r} = -\frac{\nu_1}{r} - \frac{4\pi Q^2}{r^3}.$$

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}r} = -\frac{\nu_1}{r} - \frac{4\pi Q^2}{r^3}$$

Guess a solution of the form $\nu_1 = \alpha/r^2$: then we find

$$-2\alpha = -\alpha - 4\pi Q^2$$
$$\Rightarrow \alpha = 4\pi Q^2,$$

so the full solution is

$$e^{-2\Lambda} = 1 - \frac{2GM}{r} + \frac{4\pi Q^2}{r^2}$$