Index/index free notation

$$T(X, Y, \eta, \mu) = T_{\mu\nu}{}^{\rho\sigma} X^\mu Y^\nu \eta_\rho \mu_\sigma = T_{ab}{}^{cd} X^a Y^b \eta_c \mu_d = T_{ab}{}^{cd} Y^b \eta_c X^a \mu_d$$

$$T^a{}_b = T(\mathrm{d}x^a, \partial_b)$$

$$\begin{aligned} \nabla_X Y &= X^b (\nabla_b Y^a) \partial_a \\ &\leftrightarrow X^\nu \nabla_\nu Y^\mu = X^b (\nabla_b Y^a)(\partial_a)^\mu \end{aligned}$$

2020 Q3 d)

$$L = -(1+r^2)\dot{t}^2 + (1+r^2)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2).$$

By the usual arguments we can restrict to $\theta = \frac{\pi}{2}$. Also have conserved quantities

$$\Omega = r^2 \sin^2\theta\dot{\phi} = r^2\dot{\phi}$$

$$E = (1+r^2)\dot{t}$$

$$L = -1.$$

Since we start at $r = 0$, $\Omega = 0$. Hence

$$-1 = (1+r)^{-1}(-E^2 + \dot{r}^2).$$

$$\dot{r}^2 = E^2 - 1 - r^2$$

$$\Rightarrow r = \sqrt{E^2 - 1}\sin\tau_A.$$

So the clock returns when $\tau_A = \pi$.
On the other hand, for the observer remaining at the origin, $-1 = -\dot{t}$, so $\tau_B = t$. To compare the two times, we want to express τ_A in terms of the coordinate time t.

$$E = (1 + r^2)\frac{\mathrm{d}t}{\mathrm{d}\tau_A} = (1 + (E^2 - 1)\sin^2\tau_A)\frac{\mathrm{d}t}{\mathrm{d}\tau_A}$$

$$\Rightarrow t = \int_0^{\tau_A} \left(\frac{E}{(1 + (E^2 - 1)\sin^2\tau)}\right) \mathrm{d}\tau$$

$$= \arctan(E \tan\tau_A)$$

So when $\tau_A = \pi$, we also have $\tau_B = t = \pi$!

2016 Question 1

Consider the following two-dimensional metric

$$\mathrm{d}s^2 = -cosh^2\rho \mathrm{d}t^2 + \mathrm{d}\rho^2$$

where $-\infty < t < \infty$ and $0 \leq \rho < \infty$.

(a) Write down a Lagrangian for affinely parametrized geodesics. Show that

$$\epsilon = \cosh^2 \rho \dot{t}$$

and the Lagrangian itself are conserved.

(b) Using the coordinate transformation $v = \tanh \rho$, show that

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 + v^2 = 1 - \frac{\kappa}{\epsilon^2}$$

where $\kappa = 0$ for a null geodesic, and $\kappa > 0$ for a time-like geodesic.

(Skip **(a)**) Since $v = \tanh\rho$ we have $\mathrm{d}v = \frac{1}{\cosh^2\rho}\mathrm{d}\rho$, so the metric is

$$\mathrm{d}s^2 = \cosh^2\rho(-\mathrm{d}t^2 + \cosh^2\rho\mathrm{d}v^2) = -\frac{1}{1-v^2}\mathrm{d}t^2 + \frac{1}{(1-v^2)^2}\mathrm{d}v^2)$$

Note that $\epsilon = \frac{1}{1-v^2}\dot{t}$. For affinely parametrised geodesics we have

$$-\kappa = -\frac{1}{1-v^2}\dot{t}^2 + \frac{1}{(1-v^2)^2}\dot{v}^2 = -(1-v^2)\epsilon^2 + \frac{1}{(1-v^2)^2}\dot{v}^2$$

where κ is constant (affine parameter), $\kappa = 0$ for null geodesics and $\kappa > 0$ for timelike geodesics.
Now we have

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\dot{v}}{\dot{t}} = \epsilon^{-1}\frac{1}{1-v^2}\dot{v},$$

so

$$-\kappa = -\epsilon^2(1-v^2) + \epsilon^2\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2$$

(c) Explain why $\kappa = 1$ for a time-like geodesic parametrized by proper time and hence show that $\epsilon \geq 1$ for such a geodesic. Find the time-like geodesic with $\epsilon = 1$ and hence explain the physical meaning of the coordinate t.

For timelike geodesics parametrised by proper time, $g(X, X) = -1$ where X is tangent to the geodesic. But

$$g(X, X) = -\frac{1}{1 - v^2}\dot{t}^2 + \frac{1}{(1 - v^2)^2}\dot{v}^2 = -\kappa.$$

In this case,

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 + v^2 = 1 - \frac{1}{\epsilon^2},$$

and since the LHS is ≥ 0 we must have $\epsilon^2 \geq 1$. Also, $\epsilon \geq 0$ since t increases to the future, so in fact $\epsilon \geq 1$.

If $\epsilon = 1$,

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 + v^2 = 0.$$

Hence $v = 0$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$.
In view of $\epsilon = 1$, we find that $\dot{t} = 1$, i.e. the coordinate t agrees (up to an additive constant) with the proper time τ along a stationary geodesic through $\rho = 0$.

(d) Consider geodesics starting at the origin $\rho = 0$ with $\dot{\rho} > 0$. Sketch the trajectories of both null and time-like geodesics in the (v, t)-plane. Show that a null geodesic will reach $\rho = \infty$ at an infinite value of the affine parameter, whereas a time-like geodesic will return to $\rho = 0$ after proper time π.

[You may use without proof the definite integral

$$\int_0^{\pi} \frac{\mathrm{d}t}{1 - \gamma^2 \sin^2 t} = \frac{\pi}{1 - \gamma^2}$$

which is valid for $0 \leq \gamma < 1$*.]*

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 = 1 - \frac{\kappa}{\epsilon^2} - v^2$$

$$\rightarrow t = \int \frac{\mathrm{d}v}{\sqrt{1 - \frac{\kappa}{\epsilon^2} - v^2}}$$ (while v is increasing, but symmetry arguments deal with decreasing case)

$$\Rightarrow t = \arcsin\left(v \Big/ \sqrt{1 - \frac{\kappa}{\epsilon^2}}\right)$$

$$v = \sqrt{1 - \frac{\kappa}{\epsilon^2}} \sin t.$$

Timelike case ($\kappa = 1$)

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \epsilon(1 - v^2) = \epsilon \left(1 - \left(1 - \frac{1}{\epsilon^2}\right) \sin^2 t\right)$$

$$\Rightarrow \lambda_{\text{(return)}} = \frac{1}{\epsilon} \int_0^{\pi} \frac{\mathrm{d}t}{\left(1 - \left(1 - \frac{1}{\epsilon^2}\right) \sin^2 t\right)}$$

$$= \frac{\pi}{\epsilon(1 - 1 + 1/\epsilon)} = \pi.$$

Null case: $v = \sin t$, $\frac{\mathrm{d}t}{\mathrm{d}\lambda} = \epsilon \cos^2 t$, so $\lambda = \epsilon^{-1} \tan t$, $v = \sin(\tan^{-1}(\epsilon\lambda))$.
So as $\lambda \to \infty$, $v \to 1$, so $\rho \to \infty$.

2016 Q2 d)

A stationary observer at radius r_1 (in Schwarzschild) emits a photon of frequency ω_1, which is received by a stationary observer at radius $r_2 > r_1$. Using the results from parts (a), (b) and (c), show that the frequency ω_2 measured by the observer at radius r_2 is given by

$$\frac{\omega_2}{\omega_1} = \sqrt{\frac{1 - 2m/r_1}{1 - 2m/r_2}}.$$

Comment on the frequency observed at radius r_2 as $r_1 \to 2m$.
[You may use without proof at the frequency ω of a photon with tangent vector V^a measured by a coincident observer with 4-velocity U^a is $\omega = U_a V^a$.]

Along the worldline of the photon, $(1 - 2m/r)\dot{t} = E$ is constant. So the tangent to the photon is

$$V = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\partial_a = E(1 - 2m/r)^{-1}\partial_t + V^r\partial_r + V^\theta\partial_\theta + V^\phi\partial_\phi.$$

The tangent to a stationary observer is

$$U = \frac{1}{\sqrt{1 - 2m/r}}\partial_t.$$

Hence

$$\frac{\omega_2}{\omega_1} = \frac{g(V, U_2)}{g(V, U_1)} = \sqrt{\frac{1 - 2m/r_1}{1 - 2m/r_2}}.$$

As $r_1 \to 2m$, the frequency observed by the observer at r_2 approaches zero, and the signals are infinitely redshifted.

2014 Question 1

Consider a timelike geodesic in the following two-dimensional spacetime:

$$\mathrm{d}s^2 = e^{2g\xi}(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2)$$

where $g > 0$.

(a) Show that

$$E = e^{2g\xi}\dot{\eta}$$

is conserved along the geodesic, where $\dot{\eta}$ denotes the derivative of η with respect to proper time. Show that

$$\dot{\xi}^2 = e^{-4g\xi}(E^2 - e^{2g\xi}).$$

Affinely parametrised geodesics extremise the action associated with the Lagrangian

$$L = e^{2g\xi}(-\dot{\eta}^2 + \dot{\xi}^2).$$

Since $\partial L/\partial \eta = 0$, the quantity $\partial L/\partial \dot{\eta}$ is constant, i.e. $e^{2g\xi}\dot{\eta} = E$ is constant.
Then, since τ is the proper time (or since L is independent of τ) we have $L = -1$, i.e.

$$-1 = -e^{-2g\xi}E^2 + e^{2g\xi}\dot{\xi}^2$$
$$\Rightarrow \dot{\xi}^2 = e^{-4g\xi}\left(E^2 - e^{2g\xi}\right)$$

(b) Use your results in part (a) to obtain an equation for $(\mathrm{d}\xi/\mathrm{d}\eta)^2$ and explain why an observer following a timelike geodesic who initially moves in the $+\xi$ direction will eventually turn around and approach $\xi = -\infty$.

$$\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \frac{\mathrm{d}\xi}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\eta} = \frac{\mathrm{d}\xi}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^{-1} = \frac{\dot{\xi}}{\dot{\eta}}.$$

We also have

$$\dot{\eta} = Ee^{-2g\xi},$$

hence

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)^2 = 1 - E^{-2}e^{2g\xi}$$

and we see that, if ξ is initially increasing with η, then eventually ξ will become sufficiently large that we will have $E^{-2}e^{2g\xi} = 1$, at which point $\frac{\mathrm{d}\xi}{\mathrm{d}\eta}$ will have decreased to zero. After this, $\dot{\xi}$ will become negative (since $\dot{\xi}$ is smooth).

2013 Q3

Skip part (a) – not applicable to the course now. I'll also change the signature of the metric to match our conventions.
The metric is

$$g = -\left(1 - \frac{\Lambda}{3} r^2\right) \mathrm{d}t^2 + \left(1 - \frac{\Lambda}{3} r^2\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right).$$

b) Consider geodesics in the equatorial plane, $\theta = \pi/2$. Show that

$$E = \left(1 - \frac{\Lambda}{3} r^2\right) \dot{t}, \qquad J = r^2 \dot{\phi}$$

are conserved along these geodesics, where $\dot{x}^a$ refers to the derivative of the coordinate x^a with respect to the affine parameter of the geodesic. For timelike geodesics, obtain an equation for $\dot{r}$ in terms of an effective potential. Can bound orbits occur in this spacetime? If so, compute the radius of circular orbits in terms of J and Λ. Distinguish between the cases $\Lambda > 0$ and $\Lambda < 0$.

Solution: Skip conserved quantities. Substituting conserved quantities into the equation $L = -1$ (for timelike geodesics) we get

$$-1 = \left(1 - \frac{\Lambda}{3} r^2\right)^{-1} \left(-E^2 + \dot{r}^2\right) + r^{-2} J^2$$

$$\Rightarrow \frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(-\frac{\Lambda}{3} r^2 + J^2 r^{-2}\right) = \frac{1}{2}E^2 + \frac{\Lambda}{6}J^2 - \frac{1}{2},$$

so the effective potential is

$$V(r) = \frac{1}{2}\left(-\frac{\Lambda}{3} r^2 + J^2 r^{-2}\right) \qquad \Rightarrow V'(r) = -\frac{\Lambda}{3} r - J^2 r^{-3}.$$

If $\Lambda > 0$ the V strictly decreases and there are no bound orbits. If $\Lambda < 0$ and $J \neq 0$ then V has a local minimum, so there are bound orbits.

For circular orbits, we need $\dot{r} = \ddot{r} = 0$, so V must be at a local extremum. Hence $-\frac{\Lambda}{3}r - J^2 r^{-3} + = 0$, and so

$$r = \left(\frac{3J^2}{\Lambda}\right)^{\frac{1}{4}}.$$

c) Consider null geodesics in the equatorial plane, $\theta = \pi/2$. What is the effective potential for these geodesics? Show that photons travel along straight lines in this spacetime.

Solution:

Setting $L = 0$ and substituting the conserved quantities,

$$0 = \left(1 - \frac{\Lambda}{3} r^2\right)^{-1} \left(-E^2 + \dot{r}^2\right) + r^{-2} J^2$$

$$\Rightarrow \frac{1}{2}\dot{r}^2 + \frac{1}{2} J^2 r^{-2} = \frac{1}{2} E^2 + \frac{\Lambda}{6} J^2,$$

so the effective potential is $\frac{1}{2} J^2 r^{-2}$.

Now we have

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{r^2}{J}\dot{r},$$

and so

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + r^2 = \left(\frac{E^2}{J^2} + \frac{\Lambda}{3}\right) r^4.$$

We can either solve this by integrating directly and using trig substitutions, or by guessing $rcos(\phi - \phi_0) = b$ for some constant b, which represents "straight lines" and solves the equations with impact parameter

$$b = \left(\frac{E^2}{J^2} + \frac{\Lambda}{3}\right)^{-\frac{1}{2}}.$$

d) For the case $\Lambda < 0$, consider two static observers located at radii r_1 and r_2. Suppose the observer at r_1 sends a photon to the observer at r_2. What is the ratio of the photon frequency measured by the observer at r_1 to the photon frequency measured by the observer at r_2?

Solution:

Using expressions from above, the photon frequency is $g(U, V)$, where U is the tangent vector to the observer's worldline and V is the tangent to the photon's worldline.

For static observers, $U = c\partial_t$ for some constant c. Since also $g(U, U) = -1$, we have

$$-1 = -\left(1 - \frac{\Lambda}{3}r^2\right)c^2$$

and so

$$U = \left(1 - \frac{\Lambda}{3}r^2\right)^{-\frac{1}{2}}\partial_t.$$

Along the worldline of the photon, we have the constant

$$E = \left(1 - \frac{\Lambda}{3}r^2\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda},$$

so the tangent to the photon is

$$V = E\left(1 - \frac{\Lambda}{3}r^2\right)^{-1}\partial_t + V^r\partial - r + V^\theta\partial_\theta + V^\phi\partial_\phi$$

Hence for a static observer at $r = R$,

$$\omega = g(U, V) = E\left(1 - \frac{\Lambda}{3}R^2\right)^{-\frac{1}{2}}$$

and so

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{1 - \frac{\Lambda}{3}(r_2)^2}{1 - \frac{\Lambda}{3}(r_1)^2}}.$$

e) For the case $\Lambda > 0$, the metric can be written in different coordinates as

$$g = -\mathrm{d}t'^2 + e^{2Ht'}\left((\mathrm{d}r')^2 + (r')^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right)$$

where H is a constant. Suppose that a static observer at $r' = 0$ emits a photon at $t' = 0$. What radius r' will the photon reach as $t' \to \infty$? By comparing this result to your result in part *a*, deduce how H is related to Λ.

Solution:
Cut to the chase:

$$0 = -(\dot{t}')^2 + e^{2Ht'}(\dot{r}')^2$$

$$\Rightarrow \frac{\mathrm{d}r'}{\mathrm{d}t'} = e^{-Ht'}$$

$$\Rightarrow r' = \frac{1}{H}\left(1 - e^{-Ht'}\right),$$

so $r' \to \frac{1}{H}$ as $t' \to \infty$.