Stationary solutions

Slightly vague question regarding ignoring coordinates in a stationary situation.
In general, *stationary* means that, if we choose the right coordinates, all of the components of the metric will be independent of the "time" coordinate. This can simplify some calculations: for example, the coordinate time for a light signal to travel from a place with spatial coordinates $\mathbf{x}$ to a place with spatial coordinates $\mathbf{y}$ will be *independent of the time when the light signal is sent*. On the other hand, if two observers are moving relative to the stationary coordinates, then various physical quantities *will* depend on the time at which they are measured.

On a related note: to calculate redshift, in some situations (stationary spacetime, stationary observers) we can simply take the ratio

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}t} \bigg/ \frac{\mathrm{d}\tau_A}{\mathrm{d}t}$$

but in other situations we cannot. If one or more of the observers are moving relative to the stationary coordinates, then this formula might not give the right answer. Even in Minkowski space, for two inertial observers, if their worldlines are not "parallel' then there will be some effect due to the changing times signals take to get from one observer to the other – the Doppler effect!

Symmetries of the Riemann tensor

The definition is

$$R^a{}_{bcd} X^b = (\nabla_c \nabla_d - \nabla_d \nabla_c) X^a,$$

so we immediately get antisymmetry in the last two indices,
$R^a{}_{bcd} = -R^a{}_{bdc}$
Next, apply $(\nabla_c \nabla_d - \nabla_d \nabla_c)$ to the metric tensor:

$$\begin{aligned} 0 &= (\nabla_c \nabla_d - \nabla_d \nabla_c) g_{ab} \\ &= -R^e{}_{acd} g_{eb} - R^e{}_{bcd} g_{ae} \\ \Rightarrow R_{abcd} &= -R_{bacd}, \end{aligned}$$

where we used the fact that $\nabla g = 0$.

Finally, for a scalar field f, consider

$$\begin{aligned}
&\nabla_a \nabla_b \nabla_c f + \nabla_b \nabla_c \nabla_a f + \nabla_c \nabla_a \nabla_b f \\
&- \nabla_b \nabla_a \nabla_c f - \nabla_a \nabla_c \nabla_b f - \nabla_c \nabla_b \nabla_a f \\
&= \nabla_a [\nabla_b, \nabla_c] f + \nabla_b [\nabla_c, \nabla_a] f + \nabla_c [\nabla_a, \nabla_b] f = 0 \\
&= [\nabla_a, \nabla_b] \nabla_c f + [\nabla_b, \nabla_c] \nabla_a f + [\nabla_c, \nabla_a] \nabla_b f \\
&= R^d{}_{cab} \nabla_d f + R^d{}_{abc} \nabla_d f + R^d{}_{bca} \nabla_d f \\
&= \left(R^d{}_{cab} + R^d{}_{abc} + R^d{}_{bca} \right) \nabla_d f
\end{aligned}$$

so $R^d{}_{cab} + R^d{}_{abc} + R^d{}_{bca} = 0$ (note that we can span the cotangent space with covectors of the form ∇f).

2019 Question 2 d)

General tip: a metric of the form $g = -\mathrm{d}t^2 + \mathrm{d}x^2$ can be transformed into "null" form by defining

$$u = t - x \qquad v = t + x,$$

in which case $g = -\mathrm{d}u\mathrm{d}v$.

2011 Question 3

We have reduced the null geodesic equation to

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \alpha^2 + 2u^3 - u^2,$$

where $u = m/r$, $\alpha = Em/J$.
Show that a photon may have a stable circular orbit at $r = 3m$.

Solution

For a circular orbit, we must be at a local extremum of the effective potential, $V(u) = -2u^3 + u^2$. Since $V'(u) = -6u^2 + 2u$ (and $u > 0$) there is a unique extremum at $u = 1/3$. Also, $V''(u) = -12u + 2$, so $V''(1/3) = -2 > 0$ and this is a local **maximum** (hence **unstable**!).
Finally, $u = 1/3$ means $r = 3m$.

Verify that

$$Ae^{\phi} = \frac{1-3u}{(\sqrt{3}+\sqrt{1+6u})^2}$$

is a solution to the geodesic equation, for an arbitrary constant A. Give a qualitative description of the photon orbits as we approach the limit $\phi = -\infty$ for $A = 0$ and $A > 0$.

Solution

Differentiating implicitly, we find that

$$\frac{\mathrm{d}u}{\mathrm{d}\phi}\left(\frac{-3(3+\sqrt{3}\sqrt{1+6u})}{\sqrt{1+6u}(\sqrt{3}+\sqrt{1+6u})^3}\right) = Ae^{\phi} = \frac{1-3u}{(\sqrt{3}+\sqrt{1+6u})^2}.$$

Some algebra then leads to

$$\frac{\mathrm{d}u}{\mathrm{d}\phi} = -\frac{(1-3u)\sqrt{1+6u}}{3\sqrt{3}} \qquad \Rightarrow \left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 = \frac{1}{27} - u^2 + 3u^3,$$

so this solves the required equation (with $\alpha^2 = 1/27$).
Qualitatively, if $A = 0$ we have $u = 1/3$, and so we just have the unstable circular orbit. If $A > 0$ then, as $\phi \to \infty$, we also have an orbit which approaches the circular orbit (since $u \to 1/3$). In fact, if $A > 0$ then we see that we must have $u < 1/3$, and so (recalling $u = m/r$) this corresponds to an orbit which "spirals out" from $r/3m$ as ϕ increases.

2012 Question 1 c)

Using local inertial coordinates, show that if the connection is metric compatible and torsion free, then $R_{abcd} = R_{bacd} = R_{abdc} = R_{cdab}$ and $R_{abcd} + R_{bcad} + R_{cabd} = 0$. Why is it sufficient to use local inertial coordinates to prove these identities?

Solution

We proved most of these identities already, but not using any special coordinates. In local inertial coordinates,

$$R^a{}_{bcd} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb},$$

from which we can also derive the required expressions.
(The remaining identity, $R_{abcd} = R_{cdab}$, can be obtained by taking the Bianchi identity $R_{abcd} + R_{acdb} + R_{adbc} = 0$, cyclicly permuting indices to obtain 4 such expressions, adding the first and last expressions and subtracting the middle two, and finally using antisymmetry in the first and last pairs of indices.)

Why is it sufficient to use local inertial coordinates to prove these identities?

What they want you to say: The identities in questions are "tensorial", and so, if they hold in one coordinate system then they will hold in all coordinate systems.

For example, if we have shown that $X_a = Y_a$ in some coordinate system, then $(X - Y)_a = 0$. But when we change coordinates this relationship is maintained, since $(X - Y)_{a'} = (X - Y)_a \frac{\partial x^a}{\partial y^{a'}} = 0$.

A better answer: This question is similar to the following: *suppose that, in some basis, all the components of a vector X are zero. Show that $X = 0$.* You would answer this question by saying that, expanding in the basis in question,

$$X = X^a e_a = 0$$

since $X^a = 0$ for all a.
Similarly, if we know that $X_a = Y_a$ in some coordinate system, then this means that

$$X - Y = (X_a - Y_a)\mathrm{d}x^a = 0,$$

and so $X = Y$ (and hence, although this is unnecessary to state, $X_a = Y_a$ in all coordinate systems).

2020 Question 1. c)

A satellite moves in an ingoing radial direction along a geodesic in the $r > 2M$ region of Schwarzschild spacetime. An observer, Alice, moves along a worldline where $r = R$, for some large constant $R \gg 2M$. The angular coordinates along Alice's worldline also take constant values, which are the same as those along the satellite's geodesic.

The proper time along the satellite's worldline is τ_S, while the proper time along Alice's worldline is τ_A.

(i) Assuming that $\tau_A = 0$ when $t = 0$, show that the proper time along Alice's worldline is given by

$$\tau_A = t + O\left(\frac{M}{R}\right).$$

Solution

Alice's worldline is given by

$$(t, r, \theta, \phi) = (t(\tau_A), R, \theta_0, \phi_0)$$

where θ_0 and ϕ_0 are constants. Hence the tangent to Alice's worldline is X, where

$$X = \frac{\mathrm{d}t}{\mathrm{d}\tau_A}\partial_t.$$

Since τ_A is the proper time along Alice's worldline, we must have $g(X, X) = -1$. Hence

$$\begin{aligned} -1 &= -\left(1 - \frac{2M}{R}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau_A}\right)^2 \\ \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\tau_A} &= \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} \\ &= 1 + O\left(\frac{M}{R}\right). \end{aligned}$$

Integrating this, and using the fact that $\tau_A = 0$ when $t = 0$, we obtain

$$t = \tau_A + O\left(\frac{M}{R}\right).$$

(ii) The satellite emits a (radial) light signal when $\tau_S = \tau_0$. Suppose that, at this proper time, the satellite is at the point $t = t_0$, $r = r_0$ (where $r_0 > 2M$). Show that this signal reaches Alice when $\tau_A = \tau_1$, where

$$\tau_1 = t_0 + R - r_0 + 2M \log\left(\frac{R}{r_0 - 2M}\right) + O\left(\frac{M}{R}\right).$$

Solution

Along a radial null geodesic with affine parameter λ we can use the conserved quantity E (due to the fact that the Lagrangian is independent of t) to find

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = E\left(1 - \frac{2M}{r}\right)^{-1}.$$

But also, since this is an outgoing radial null geodesic, we have

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = E.$$

Putting these two together, we have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 1 - \frac{2M}{r}$$

$$\Rightarrow \mathrm{d}t = \left(1 + \frac{2M}{r - 2M}\right)\mathrm{d}r.$$

Integrating from $r = r_0$ (when $t = t_0$) to $r = R$ (when $t = t_1$) we obtain

$$t_1 - t_0 = R - r_0 + 2M \log\left(\frac{R - 2M}{r_0 - 2M}\right).$$

Now, since the proper time measured by Alice matches the coordinate time to leading order in $\frac{M}{R}$, we have

$$\begin{aligned}
\tau_1 &= t_0 + R - r_0 + 2M \log\left(\frac{R - 2M}{r_0 - 2M}\right) + O\left(\frac{M}{R}\right) \\
&= t_0 + R - r_0 + 2M \log\left(\frac{R}{r_0 - 2M}\right) + 2M \log\left(\frac{R - 2M}{R}\right) + O\left(\frac{M}{R}\right) \\
&= t_0 + R - r_0 + 2M \log\left(\frac{R}{r_0 - 2M}\right) + 2M \log\left(1 - \frac{2M}{R}\right) + O\left(\frac{M}{R}\right) \\
&= t_0 + R - r_0 + 2M \log\left(\frac{R}{r_0 - 2M}\right) + O\left(\frac{M}{R}\right).
\end{aligned}$$

d) (i) The satellite emits a second light signal when $\tau_S = \tau_0 + \Delta\tau_S$. This signal is received by Alice when $\tau_A = \tau_1 + \Delta\tau_A$. Neglecting terms of order $(\Delta\tau_S)^2$ and terms of order M/R, show that, if the energy of the geodesic on which the satellite moves is E, then

$$\Delta\tau_A = \frac{Er_0}{r_0 - 2M}\left(1 + \sqrt{1 - E^{-2}\left(\frac{r_0 - 2M}{r_0}\right)}\right)\Delta\tau_S + O\left((\Delta\tau_S)^2\right) + O\left(\frac{M}{R}\right).$$

Solution

The satellite emits the second signal at $t = t(\tau_0 + \Delta\tau_S)$, $r = r(\tau_0 + \Delta\tau_S)$, where these are the points along the worldline of the satellite, parametrised by its proper time τ_S.
Since we are neglecting terms of order $(\Delta\tau_S)^2$, we have

$$t(\tau_0 + \Delta\tau_S) = t_0 + \frac{\mathrm{d}t}{\mathrm{d}\tau_s}\Delta\tau_S + O\left((\Delta\tau_S)^2\right)$$
$$r(\tau_0 + \Delta\tau_S) = r_0 + \frac{\mathrm{d}r}{\mathrm{d}\tau_s}\Delta\tau_S + O\left((\Delta\tau_S)^2\right).$$

Furthermore, we have

$$\tau_1 + \Delta\tau_1 = \tau_1(\tau_0 + \Delta\tau_0)$$
$$\Rightarrow \Delta\tau_1 = \frac{\mathrm{d}\tau_1}{\mathrm{d}\tau_0}\Delta\tau_0 + O\left((\Delta\tau_S)^2\right),$$

where we are writing $\tau_1 = \tau_1\left(r_0(\tau_0), t_0(\tau_0)\right)$.

Differentiating the formula obtained in the previous part (r_0 and t_0 are now functions of τ_S), we find

$$\Delta\tau_A = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau_S} - \frac{\mathrm{d}r}{\mathrm{d}\tau_S} - \frac{2M}{r_0 - 2M}\frac{\mathrm{d}r}{\mathrm{d}\tau_S}\right)\Delta\tau_S + O\left((\Delta\tau_S)^2\right) + O\left(\frac{M}{R}\right) \quad (1)$$

where the various terms are evaluated at $r = r_0$, $t = t_0$.
Along the worldline of the satellite, the conserved energy gives us

$$\frac{\mathrm{d}t}{\mathrm{d}\tau_S} = E\left(1 - \frac{2M}{r}\right)^{-1}$$

so at $r = r_0$,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau_S} = E\left(\frac{r_0}{r_0 - 2M}\right). \quad (2)$$

Next, since the satellite moves along an ingoing radial timelike geodesic parametrised by proper time τ_S, we have

$$-1 = \left(1 - \frac{2M}{r}\right)^{-1}\left(-E^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau_S}\right)^2\right)$$

$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\tau_S} = E\sqrt{1 - E^{-2}\left(\frac{r - 2M}{r}\right)}.$$

Evaluating this at $r = r_0$ and substituting the result, along with equation (2) into equation (1), we obtain

$$\Delta\tau_A = \frac{Er_0}{r_0 - 2M}\left(1 + \sqrt{1 - E^{-2}\left(\frac{r_0 - 2M}{r_0}\right)}\right)\Delta\tau_S + O\left((\Delta\tau_S)^2\right) + O\left(\frac{M}{R}\right).$$

(ii) What happens to the satellite when $r_0 = 2M$? What happens to the signals received by Alice when this occurs?

Solution

$$\Delta\tau_A = \frac{Er_0}{r_0 - 2M}\left(1 + \sqrt{1 - E^{-2}\left(\frac{r_0 - 2M}{r_0}\right)}\right)\Delta\tau_S + O\left((\Delta\tau_S)^2\right) + O\left(\frac{M}{R}\right).$$

When $r_0 = 2M$ the satellite reaches the event horizon of the black hole. At this point, $\Delta\tau_A \to \infty$, so the frequency of the signals received by Alice tends to infinity – they are infinitely redshifted.

2013 Question 1

(a) State the weak and strong versions of the Principle of Equivalence.

Weak – the trajectory of a point mass in a gravitational field depends only on its initial position and velocity.

Strong: The gravitational motion of a small test body depends only on its initial position in spacetime and velocity, and not on its constitution, *and* the outcome of any local experiment (gravitational or not) in a freely falling laboratory is independent of the velocity of the laboratory and its location in spacetime.

(b) Consider the metric for 3-dimensional Minkowski space (t', r', ϕ') where t' is the time coordinate, and (r', ϕ') are 2-dimensional polar coordinates:

$$\mathrm{d}s^2 = -(\mathrm{d}t')^2 + (\mathrm{d}r')^2 + (r')^2(\mathrm{d}\phi')^2.$$

Now consider a coordinate transformation to a frame which is rotating with constant angular velocity ω:

$$t' = t, \quad r' = r, \quad \phi' = \phi + \omega t.$$

What is the metric in the rotating frame (t, r, ϕ)?

We have

$$\mathrm{d}t' = \mathrm{d}t \qquad \mathrm{d}r' = \mathrm{d}r \qquad \mathrm{d}\phi' = \mathrm{d}\phi + \omega \mathrm{d}t,$$

so the line element is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\phi + \omega \mathrm{d}t\right)^2$$

$$= -(1 - r^2\omega^2)\mathrm{d}t^2 + 2\omega r^2 \mathrm{d}t\mathrm{d}\phi + \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2$$

(c) Consider a static observer in the rotating frame who is located at the position $(r, \phi) = (R, 0)$. How is the proper time of this observer related to the coordinate time t? Explain the physical significance of this result.

Since this observer moves along the curve $(t, r, \phi) = (t(\tau), R, 0)$, the tangent to this curve is $\dot{t}\partial_t$. Since τ is proper time, this vector has norm -1, i.e. $\dot{t}^2 g_{tt} = -1$, which means

$$-(1 - R^2\omega^2)\dot{t}^2 = -1 \qquad \Rightarrow \quad \dot{t} = (1 - R^2\omega^2)^{-\frac{1}{2}}$$

where we took the positive root because t increases as proper time increases. Hence

$$t - t_0 = (1 - R^2\omega^2)^{-\frac{1}{2}}\tau$$

where t_0 is constant.
Hence time passes more slowly for the rotating observer relative an observer at the origin (who measures proper time t).

(d) Compute the 4-acceleration of the static observer in part (c) in (t, r, ϕ) coordinates, where the 4-acceleration is defined to be

$$a^\mu = u^\nu \nabla_\nu u^\mu$$

and u^μ is the 4-velocity of the observer. Explain the physical significance of this result.

We have

$$\begin{aligned}
a^\mu = u^\nu \nabla_\nu u^\mu &= (1 - R^2\omega^2)^{-\frac{1}{2}} \left(\partial_t u^\mu + \Gamma^\mu_{t\rho} u^\rho \right) = (1 - R^2\omega^2)^{-1} \Gamma^\mu_{tt} \\
&= \frac{1}{2}(1 - R^2\omega^2)^{-1} g^{\mu\nu} \left(2\partial_t g_{t\nu} - \partial_\nu g_{tt} \right) \\
&= -\frac{1}{2}(1 - R^2\omega^2)^{-1} g^{\mu r} \left(\partial_r g_{tt} \right) \\
&= -\frac{R\omega^2}{1 - R^2\omega^2} \delta^\mu_r ,
\end{aligned}$$

so the observer accelerates towards the centre as expected.

(e) For a static metric $g_{\mu\nu}$ which is independent of the coordinate time t, the metric for a $t =$ constant spatial hypersurface as measured by static observers is given by

$$\gamma_{ij} = g_{ij} - \frac{g_{ti} g_{tj}}{g_{tt}}.$$

Using the metric you obtained in part (b), compute the spatial metric $\gamma_{ij}$for observers rotating with angular velocity ω in 3-dimensional Minkowski space. Using γ_{ij} , compute the circumference of a circle of radius R as measured by these observers. Explain the physical significance of this result.

We have

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \frac{\omega^2 r^4}{(1-r^2\omega^2)}\mathrm{d}\phi^2 = \mathrm{d}r^2 + \frac{r^2}{(1-r^2\omega^2)}\mathrm{d}\phi^2,$$

so the circumference of a circle of radius R measured by these observers is $2\pi\frac{R}{\sqrt{1-R^2\omega^2}}$. Hence the geometry of the spatial slices measured by rotating observers is non-Euclidean!

2015 Question 3

Consider the Schwarzschild line element,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

(a) Explain why we can take geodesics to lie in the equatorial plane ($\theta = \pi/2$) without loss of generality. Then show that a null geodesic in the equatorial plane obeys

$$\dot{r}^2 = E^2 - \left(1 - \frac{2M}{r}\right)\frac{J^2}{r^2}$$

$$\text{where } E = \left(1 - \frac{2M}{r}\right)\dot{t} \quad , \quad J = r^2\dot{\phi},$$

and show that E and J are constants of motion.

Consider a geodesic through the point p in the Schwarzschild geometry, with tangent vector X at p. Then, since the metric is invariant under rotations, we can perform a rotation so that p lies in the equatorial plane, and then another rotation (if necessary) which fixes p and rotates X so that X is tangent to the equatorial plane. Hence, without loss of generality, we can suppose that *initially* $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$. But the equation of motion for θ is

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\dot{\theta}\right) - r^2 \sin\theta \cos\theta \dot{\phi}^2 = 0.$$

This is a second order ODE for $\theta(s)$. $\theta(s) = \frac{\pi}{2}$ solves this ODE and has the required initial data, hence it is the unique solution, and so the geodesic will remain in the equatorial plane for all time.

Affinely parametrised null geodesics in the equatorial plane extremise the action associated with the Lagrangian

$$L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2.$$

Since the Lagrangian is independent of t, $\frac{\partial L}{\partial \dot{t}}$ is constant, so E is constant. Similarly, the Lagrangian is independent of ϕ, so J is constant. Finally, the Lagrangian satisfies $L = 0$ for affinely parametrised null geodesics. Substituting for E and J, we find that

$$0 = -\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^{-2}J^2$$

$$\Rightarrow \dot{r}^2 = E^2 - \left(1 - \frac{2M}{r}\right)\frac{J^2}{r^2}$$

(b) Describe briefly what types of geodesics are allowed. If a circular orbit is possible, determine the value of its radius and comment on its stability.

We can write the equation derived above as $\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2$, with $V(r) = (1 - 2M/r)J^2/(2r^2)$. This describes a particle with unit mass and total energy $\frac{1}{2}E^2$ moving in a potential $V(r)$.

Considering this potential V, we see that, if $J = 0$ then light rays move in straight lines in the radial direction. Otherwise we see that V has a maximum at $r = 3M$ (when $V = \frac{J^2}{54M^2}$) and decreases like J^2/r^2 at large r.

Circular null geodesics are only possible at $r = 3M$. They are unstable since this is a local maximum of the potential. Otherwise, if $E^2 > J^2/(27M^2)$, then null geodesics can come in from infinity, pass over the maximum of the potential and fall into the black hole. On the other hand, if $E^2 < J^2/(27M^2)$, then null geodesics can come in towards $r = 0$ before being reflected off the potential, and then heading out in the direction of increasing r.

(c) Consider now the propagation of an electromagnetic signal between two satellites, according to the figure *(the figure shows a null geodesic travelling between satellites at radii r_1 and r_2)*. The point $r = r_0$ is the point of closest approach to the Sun for the signal propagation.
Using the geodesic equation from problem (a), find the relation between the ratio E/J and r_0. Then, rewrite that equation in terms of dr/dt and r_0, instead of $\dot{r}$, E and J.
Finally, determine the coordinate time Δt required for the propagation of the signal between the two satellites, in terms of r_0, r_1 and r_2. It is enough to specify Δt in terms of an integral expression, which you need not attempt to solve in general. Solve it only in the flat spacetime case $M = 0$, and interpret that result.

Recall

$$\dot{r}^2 = E^2 - \left(1 - \frac{2M}{r}\right)\frac{J^2}{r^2}.$$

At the point of closest approach, $\dot{r} = 0$ and $r = r_0$, so

$$E^2 = \left(1 - \frac{2M}{r_0}\right)\frac{J^2}{r_0^2} \quad \Rightarrow \quad \frac{E}{J} = \pm\frac{1}{r_0}\sqrt{1 - \frac{2M}{r_0}}$$

Now we note that $\mathrm{d}r/\mathrm{d}t = \dot{r}/\dot{t} = \frac{\dot{r}}{E}\left(1 - \frac{2M}{r}\right)$. Hence the equation for $\mathrm{d}r/\mathrm{d}t$ is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)^3\frac{J^2}{E^2}$$

$$= \left(1 - \frac{2M}{r}\right)^2\left(1 - \frac{r_0^3}{r^3}\left(\frac{r - 2M}{r_0 - 2M}\right)\right),$$

and so we see that

$$\Delta t = \int_{r_0}^{r_1}\left(1 - \frac{2M}{r}\right)^{-1}\left(1 - \frac{r_0^3}{r^3}\left(\frac{r - 2M}{r_0 - 2M}\right)\right)^{-\frac{1}{2}}\mathrm{d}r$$

$$+ \int_{r_0}^{r_2}\left(1 - \frac{2M}{r}\right)^{-1}\left(1 - \frac{r_0^3}{r^3}\left(\frac{r - 2M}{r_0 - 2M}\right)\right)^{-\frac{1}{2}}\mathrm{d}r$$

In the case $M = 0$ we find that

$$\Delta t = \int_{r_0}^{r_1} \frac{r}{\sqrt{r^2 - r_0^2}} \mathrm{d}r + \int_{r_0}^{r_2} \frac{r}{\sqrt{r^2 - r_0^2}} \mathrm{d}r = \sqrt{r_1^2 - r_0^2} + \sqrt{r_2^2 - r_0^2}$$

which is the standard formula from trigonometry, if we assume that the light ray travels in a straight line.