Covariant derivatives and notation

The notation $\nabla_a X^b$ means the (a, b) component of the $(1, 1)$ tensor (field) ∇X. Annoyingly, we might *sometimes* want to take a derivative of the *scalar field* X^a (the a-th component of X with respect to some chosen coordinate system), and this is, in general, not the same thing. The only good notation for this is $\partial_b X^a$. For example,

$$
\begin{aligned}
\nabla_X Y &= X^a \nabla_a (Y^b \partial_b) \\
&= X^a (\partial_a Y^b) \partial_b + X^a Y^b (\nabla_a \partial_b) \\
&= X^a (\partial_a Y^b) \partial_b + X^a Y^b \Gamma^c_{ab} \partial_c \\
&= X^a \left(\partial_a Y^b + \Gamma^b_{ac} Y^c \right) \partial_b \\
&= X^a (\nabla_a Y^b) \partial_b
\end{aligned}
$$

On the other hand, in the following derivation we are expanding $\nabla\eta$ using the Leibniz rule for ∇, and so when we write $\nabla_a \eta_b$ we mean (as always) the (a, b) component of the $(0, 2)$ tensor (field) $\nabla\eta$, *not* the operator ∇_a applied to the b-th component of η! (If we had meant this, we would have simply written $\partial_a \eta_b$).

$$\begin{aligned}
(\partial_a \eta_b) X^b + \eta_b (\partial_a X^b) &= \mathrm{d}_a (\eta_b X^b) \\
&= \nabla_a (\eta_b X^b) \\
&= (\nabla_a \eta_b) X^b + \eta_b (\nabla_a X^b) \\
&= (\nabla_a \eta_b) X^b + \eta_b (\partial_a X^b + \Gamma^b_{ac} X^c).
\end{aligned}$$

An alternative way to keep everything clear would be to write ${(\nabla X)_a}^b$ for the (a, b)-th component of ∇X, and to write $\nabla_a X^b$ to mean the a-th component of ∇X^b (where, since we have picked coordinates, X^b is simply a scalar function for each choice of b).

Unfortunately this is not the standard notation!

2021 Question 1 b)

A satellite orbits the Earth in a circular orbit at a radius r_S (in Schwarzschild coordinates), and Alice stands on the surface of the Earth, which has radius r_A. No external forces act on the satellite. Each time the satellite passes directly above Alice, it emits a radial light ray, which is then received by Alice.
Let the proper time along the worldline of the satellite be τ_S, and let the proper time along Alice's worldline be τ_A. At the moment when the satellite first emits a light ray, its clock reads $\tau_S = 0$, and when Alice first receives the light ray, her clock reads $\tau_A = 0$. Subsequent signals are emitted by the satellite when its clock reads $\tau_S = T_{(S,n)}$ (for $n \in \mathbb{N}$), and received by Alice when her clock reads $\tau_A = T_{(A,n)}$.

(i) For fixed $n \geq 1$, find $T_{(A,n)}$ as a function of $T_{(S,n)}$, r_S and r_A, justifying all of your calculations.

Solution:
We will calculate τ_S and τ_A as functions of the Schwarzschild time coordinate t. Then, let the Schwarzschild time for a radial light ray to go from r_S to r_A be t_0 (in the end we won't have to calculate this constant).
First we calculate $\tau_S(t)$. The worldline of the satellite, written in Schwarzschild coordinates, is $(t(\tau_S), r(\tau_S), \theta(\tau_S), \phi(\tau_S))$. This worldline extremises the Lagrangian

$$L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2,$$

where 'dots' are derivatives with respect to τ_S.

First we note that, without loss of generality, we can take the worldline of the satellite to lie in the equatorial plane $\theta = \frac{\pi}{2}$. This is because the Euler-Lagrange equation for θ is

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2 \sin\theta\cos\theta\dot{\phi} = 0,$$

and $\theta \equiv \frac{\pi}{2}$ is a solution to this ODE. Hence, if the initial conditions are $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, then this is the (unique) solution, and these initial conditions can be obtained by using the isometries associated with spherical symmetry.

Next we note that, since the Lagrangian is independent of t, E is constant, where

$$E = -\frac{1}{2}\frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{2M}{r}\right)\dot{t}.$$

Similarly, since the Lagrangian is independent of ϕ, Ω is constant, where

$$\Omega = \frac{1}{2}\frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2\theta\dot{\phi}.$$

Finally, since τ_S is the proper time along the worldline of the satellite, $L = -1$.

Putting this together, we find that

$$-1 = -\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^{-2}\Omega^2$$

$$\Rightarrow \frac{1}{2}\dot{r}^2 - \frac{M}{r} + \frac{\Omega^2}{2r^2} - \frac{M\Omega^2}{r^3} = \frac{1}{2}(E^2 - 1),$$

so the motion corresponds to the motion of a particle in one dimension, with potential $V(r) = -\frac{M}{r} + \frac{\Omega^2}{2r^2} - \frac{M\Omega^2}{r^3}$ and energy $\frac{1}{2}(E^2 - 1)$.

For a circular orbit we must have $\dot{r} = \ddot{r} = 0$, meaning that we must be at a local extremum of the potential energy (since $\ddot{r} = -V'(r)$). Since the satellite orbits at a radius r_S, the extrema of V are at r_S, where

$$Mr_S^2 - \Omega^2 r_S + 3M\Omega^2 = 0.$$

Hence the angular momentum of the satellite is given by

$$\Omega^2 = \frac{Mr_S^2}{r_S - 3M}.$$

Next, since $\dot{r} = 0$ the energy of the orbit is given by

$$E^2 = \left(1 - \frac{2M}{r_S}\right)\left(1 + r_S^{-2}\Omega^2\right) = \frac{(r_S - 2M)^2}{r_S(r_S - 3M)}.$$

Finally, recall that $\dot{t} = \left(1 - \frac{2M}{r_S}\right)^{-1} E > 0$. Hence

$$\frac{\mathrm{d}t}{\mathrm{d}\tau_S} = \sqrt{\frac{r_S}{r_S - 3M}}.$$

Hence we have

$$\tau_S = C_S + \sqrt{\frac{r_S - 3M}{r_S}}\, t,$$

for some constant C_S.

Next we compute the proper time along the worldline of Alice. Along Alice's worldline, all of the spatial coordinates are constant, and so we have

$$-1 = g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau_A}\frac{\mathrm{d}x^b}{\mathrm{d}\tau_A} = -\left(1 - \frac{2M}{r_A}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau_A}\right)^2.$$

Hence we have

$$\tau_A = C_A + \sqrt{\frac{r_A - 2M}{r_A}}\,t,$$

for some constant C_A.
Now, since the coordinate time for a radial light ray to reach Alice from Bob is t_0, which is a constant independent of n, we find that

$$T_{(A,n)} = \sqrt{\frac{r_S(r_A - 2M)}{r_A(r_S - 3M)}}\,T_{(S,n)}.$$

2020 Question 1. c)

A satellite moves in an ingoing radial direction along a geodesic in the $r > 2M$ region of Schwarzschild spacetime. An observer, Alice, moves along a worldline where $r = R$, for some large constant $R \gg 2M$. The angular coordinates along Alice's worldline also take constant values, which are the same as those along the satellite's geodesic.

The proper time along the satellite's worldline is τ_S, while the proper time along Alice's worldline is τ_A.

(i) Assuming that $\tau_A = 0$ when $t = 0$, show that the proper time along Alice's worldline is given by

$$\tau_A = t + O\left(\frac{M}{R}\right).$$

Solution

Alice's worldline is given by

$$(t, r, \theta, \phi) = (t(\tau_A), R, \theta_0, \phi_0)$$

where θ_0 and ϕ_0 are constants. Hence the tangent to Alice's worldline is X, where

$$X = \frac{\mathrm{d}t}{\mathrm{d}\tau_A}\partial_t.$$

Since τ_A is the proper time along Alice's worldline, we must have $g(X, X) = -1$. Hence

$$\begin{aligned}
-1 &= -\left(1 - \frac{2M}{R}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau_A}\right)^2 \\
\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\tau_A} &= \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} \\
&= 1 + O\left(\frac{M}{R}\right).
\end{aligned}$$

Integrating this, and using the fact that $\tau_A = 0$ when $t = 0$, we obtain

$$t = \tau_A + O\left(\frac{M}{R}\right).$$

(ii) The satellite emits a (radial) light signal when $\tau_S = \tau_0$. Suppose that, at this proper time, the satellite is at the point $t = t_0$, $r = r_0$ (where $r_0 > 2M$). Show that this signal reaches Alice when $\tau_A = \tau_1$, where

$$\tau_1 = t_0 + R - r_0 + 2M \log\left(\frac{R}{r_0 - 2M}\right) + O\left(\frac{M}{R}\right).$$

Solution

Along a radial null geodesic with affine parameter λ we can use the conserved quantity E (due to the fact that the Lagrangian is independent of t) to find

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = E\left(1 - \frac{2M}{r}\right)^{-1}.$$

But also, since this is an outgoing radial null geodesic, we have

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = E.$$

Putting these two together, we have

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= 1 - \frac{2M}{r} \\ \Rightarrow \mathrm{d}t &= \left(1 + \frac{2M}{r - 2M}\right)\mathrm{d}r. \end{aligned}$$

Integrating from $r = r_0$ (when $t = t_0$) to $r = R$ (when $t = t_1$) we obtain

$$t_1 - t_0 = R - r_0 + 2M \log\left(\frac{R - 2M}{r_0 - 2M}\right).$$

Now, since the proper time measured by Alice matches the coordinate time to leading order in $\frac{M}{R}$, we have

$$\begin{aligned}
\tau_1 &= t_0 + R - r_0 + 2M \log\left(\frac{R - 2M}{r_0 - 2M}\right) + O\left(\frac{M}{R}\right) \\
&= t_0 + R - r_0 + 2M \log\left(\frac{R}{r_0 - 2M}\right) + 2M \log\left(\frac{R - 2M}{R}\right) + O\left(\frac{M}{R}\right) \\
&= t_0 + R - r_0 + 2M \log\left(\frac{R}{r_0 - 2M}\right) + 2M \log\left(1 - \frac{2M}{R}\right) + O\left(\frac{M}{R}\right) \\
&= t_0 + R - r_0 + 2M \log\left(\frac{R}{r_0 - 2M}\right) + O\left(\frac{M}{R}\right).
\end{aligned}$$

d) (i) The satellite emits a second light signal when $\tau_S = \tau_0 + \Delta\tau_S$. This signal is received by Alice when $\tau_A = \tau_1 + \Delta\tau_A$. Neglecting terms of order $(\Delta\tau_S)^2$ and terms of order M/R, show that, if the energy of the geodesic on which the satellite moves is E, then

$$\Delta\tau_A = \frac{Er_0}{r_0 - 2M}\left(1 + \sqrt{1 - E^{-2}\left(\frac{r_0 - 2M}{r_0}\right)}\right)\Delta\tau_S + O\left((\Delta\tau_S)^2\right) + O\left(\frac{M}{R}\right).$$

Solution

The satellite emits the second signal at $t = t(\tau_0 + \Delta\tau_S)$, $r = r(\tau_0 + \Delta\tau_S)$, where these are the points along the worldline of the satellite, parametrised by its proper time τ_S.
Since we are neglecting terms of order $(\Delta\tau_S)^2$, we have

$$
\begin{aligned}
t(\tau_0 + \Delta\tau_S) &= t_0 + \frac{\mathrm{d}t}{\mathrm{d}\tau_s}\Delta\tau_S + O\left((\Delta\tau_S)^2\right) \\
r(\tau_0 + \Delta\tau_S) &= r_0 + \frac{\mathrm{d}r}{\mathrm{d}\tau_s}\Delta\tau_S + O\left((\Delta\tau_S)^2\right).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\tau_1 + \Delta\tau_1 &= \tau_1(\tau_0 + \Delta\tau_0) \\
\Rightarrow \Delta\tau_1 &= \frac{\mathrm{d}\tau_1}{\mathrm{d}\tau_0}\Delta\tau_0 + O\left((\Delta\tau_S)^2\right),
\end{aligned}
$$

where we are writing $\tau_1 = \tau_1\left(r_0(\tau_0), t_0(\tau_0)\right)$.

Differentiating the formula obtained in the previous part, we find

$$\Delta\tau_A = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau_S} - \frac{\mathrm{d}r}{\mathrm{d}\tau_S} - \frac{2M}{r_0 - 2M}\frac{\mathrm{d}r}{\mathrm{d}\tau_S}\right)\Delta\tau_S + O\left((\Delta\tau_S)^2\right) + O\left(\frac{M}{R}\right) \quad (1)$$

where the various terms are evaluated at $r = r_0$, $t = t_0$.
Along the worldline of the satellite, the conserved energy gives us

$$\frac{\mathrm{d}t}{\mathrm{d}\tau_S} = E\left(1 - \frac{2M}{r}\right)^{-1}$$

so at $r = r_0$,

$$\frac{\mathrm{d}t}{\mathrm{d}\tau_S} = E\left(\frac{r_0}{r_0 - 2M}\right). \quad (2)$$

Next, since the satellite moves along an ingoing radial timelike geodesic parametrised by proper time τ_S, we have

$$-1 = \left(1 - \frac{2M}{r}\right)^{-1}\left(-E^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau_S}\right)^2\right)$$

$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\tau_S} = E\sqrt{1 - E^{-2}\left(\frac{r - 2M}{r}\right)}.$$

Evaluating this at $r = r_0$ and substituting the result, along with equation (2) into equation (1), we obtain

$$\Delta\tau_A = \frac{Er_0}{r_0 - 2M}\left(1 + \sqrt{1 - E^{-2}\left(\frac{r_0 - 2M}{r_0}\right)}\right)\Delta\tau_S + O\left((\Delta\tau_S)^2\right) + O\left(\frac{M}{R}\right).$$

(ii) What happens to the satellite when $r_0 = 2M$? What happens to the signals received by Alice when this occurs?

Solution

$$\Delta\tau_A = \frac{Er_0}{r_0 - 2M}\left(1 + \sqrt{1 - E^{-2}\left(\frac{r_0 - 2M}{r_0}\right)}\right)\Delta\tau_S + O\left((\Delta\tau_S)^2\right) + O\left(\frac{M}{R}\right).$$

When $r_0 = M$ the satellite reaches the event horizon of the black hole. At this point, $\Delta\tau_A \to \infty$, so the frequency of the signals received by Alice tends to infinity – they are infinitely redshifted.

2014 Question 1

Consider a timelike geodesic in the following two-dimensional spacetime:

$$\mathrm{d}s^2 = e^{2g\xi}(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2)$$

where $g > 0$.
(a) Show that

$$E = e^{2g\xi}\dot{\eta}$$

is conserved along the geodesic, where $\dot{\eta}$ denotes the derivative of η with respect to proper time. Show that

$$\dot{\xi}^2 = e^{-4g\xi}(E^2 - e^{2g\xi}).$$

Affinely parametrised geodesics extremise the action associated with the Lagrangian

$$L = e^{2g\xi}(-\dot{\eta}^2 + \dot{\xi}^2).$$

Since $\partial L/\partial \eta = 0$, the quantity $\partial L/\partial \dot{\eta}$ is constant, i.e. $e^{2g\xi}\dot{\eta} = E$ is constant.
Then, since τ is the proper time (or since L is independent of τ) we have $L = -1$, i.e.

$$-1 = -e^{-2g\xi}E^2 + e^{2g\xi}\dot{\xi}^2$$
$$\Rightarrow \dot{\xi}^2 = e^{-4g\xi}\left(E^2 - e^{2g\xi}\right)$$

(b) Use your results in part (a) to obtain an equation for $(\mathrm{d}\xi/\mathrm{d}\eta)^2$ and explain why an observer following a timelike geodesic who initially moves in the $+\xi$ direction will eventually turn around and approach $\xi = -\infty$.

$$\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \frac{\mathrm{d}\xi}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\eta} = \frac{\mathrm{d}\xi}{\mathrm{d}\tau}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^{-1} = \frac{\dot{\xi}}{\dot{\eta}}.$$

We also have

$$\dot{\eta} = Ee^{-2g\xi},$$

hence

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)^2 = 1 - E^{-2}e^{2g\xi}$$

and we see that, if ξ is initially increasing with η, then eventually ξ will become sufficiently large that we will have $E^{-2}e^{2g\xi} = 1$, at which point $\frac{\mathrm{d}\xi}{\mathrm{d}\eta}$ will have decreased to zero. After this, $\dot{\xi}$ will become negative (since $\dot{\xi}$ is smooth).

(c) Consider an observer following a timelike geodesic with $E = 1$. Find the trajectory $\xi(\eta)$ which corresponds to the observer coming in from $\xi = -\infty$, turning around when $\eta = 0$, and going back out to $\xi = -\infty$.

[The integral

$$\int \frac{\mathrm{d}x}{\sqrt{1 - e^{\alpha x}}} = -\frac{2}{\alpha} \tanh^{-1} \sqrt{1 - e^{\alpha x}} + \textit{constant}$$

may be useful.]

Recall

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)^2 = 1 - E^{-2}e^{2g\xi}.$$

Set $E = 1$. While the geodesic is heading inward we have $\frac{\mathrm{d}\xi}{\mathrm{d}\eta} > 0$, so we take the positive root:

$$\frac{\mathrm{d}\xi}{\mathrm{d}\eta} = \sqrt{1 - e^{2g\xi}}.$$

Integrating this

$$\int \mathrm{d}\eta = \int \frac{\mathrm{d}\xi}{\sqrt{1 - e^{2g\xi}}}$$

and using the hint

$$\eta = -\frac{1}{g}\tanh^{-1}\sqrt{1 - e^{2g\xi}} + \eta_0.$$

We are told that $\mathrm{d}\xi/\mathrm{d}\eta = 0$ when $\eta = 0$, so $\xi = 0$ when $\eta = 0$. Hence $\eta_0 = 0$. We can invert the relationship above to find

$$\xi = -\frac{1}{g}\log\left(\cosh(g\eta)\right).$$

Same formula holds for $\eta > 0$ by symmetry.

(d) Let

$$t = \frac{1}{g} e^{g\xi} \sinh(g\eta), \qquad x = \frac{1}{g} e^{g\xi} \cosh(g\eta).$$

What is the metric in (t, x) coordinates? What is the trajectory that you computed in part (c) in (t, x) coordinates?
[The identities

$$\cosh\left(\tanh^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}, \qquad \sinh\left(\tanh^{-1} x\right) = \frac{x}{\sqrt{1-x^2}}$$

may be useful.]

We compute

$$\mathrm{d}t = e^{g\xi}\sinh(g\eta)\mathrm{d}\xi + e^{g\xi}\cosh(g\eta)\mathrm{d}\eta$$
$$\mathrm{d}x = e^{g\xi}\cosh(g\eta)\mathrm{d}\xi + e^{g\xi}\sinh(g\eta)\mathrm{d}\eta,$$

from which we see that

$$-\mathrm{d}t^2 + \mathrm{d}x^2 = e^{2g\xi}\left(-\mathrm{d}\eta^2 + \mathrm{d}\xi^2\right) = \mathrm{d}s^2$$

Now recalling

$$\xi = -\frac{1}{g}\log\left(\cosh(g\eta)\right),$$

we find that, if we parametrise this curve by η, then

$$t = \frac{1}{g}e^{-\log(\cosh(g\eta))}\sinh(g\eta) = \frac{1}{g}\tanh(g\eta)$$
$$x = \frac{1}{g}e^{-\log(\cosh(g\eta))}\cosh(g\eta) = \frac{1}{g},$$

so the observer is at rest at $x = 1/g$.

(e) Explain the physical significance of proper time. What is the proper time of the trajectory that you computed in part (c)?

Proper time along a timelike curve is the time that would be measured by an accurate clock that travels along the curve.
The path computed in part (c), parametrised by proper time τ, is given in (t, r) coordinates by

$$(t(\tau), x(\tau)) = (t(\tau), 1/g).$$

Hence the tangent to this curve is $(\mathrm{d}t/\mathrm{d}\tau)\partial_t$. Since τ is proper time this should have length -1, i.e.

$$-1 = -\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

so $\tau = \pm t + \tau_0$, for some constant τ_0. Since t increases to the future we take the positive sign. In terms of the original coordinates,

$$\tau = \frac{1}{g} e^{g\xi} \sinh(g\eta) + \tau_0.$$

2016 Question 1

Consider the following two-dimensional metric

$$\mathrm{d}s^2 = -cosh^2\rho \mathrm{d}t^2 + \mathrm{d}\rho^2$$

where $-\infty < t < \infty$ and $0 \leq \rho < \infty$.
(a) Write down a Lagrangian for affinely parametrized geodesics.
Show that

$$\epsilon = \cosh^2\rho \dot{t}$$

and the Lagrangian itself are conserved.
(b) Using the coordinate transformation $v = \tanh\rho$, show that

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 + v^2 = 1 - \frac{\kappa}{\epsilon^2}$$

where $\kappa = 0$ for a null geodesic, and $\kappa > 0$ for a time-like geodesic.

(Skip **(a)**) Since $v = \tanh\rho$ we have $\mathrm{d}v = \frac{1}{\cosh^2\rho}\mathrm{d}\rho$, so the metric is

$$\mathrm{d}s^2 = \cosh^2\rho(-\mathrm{d}t^2 + \cosh^2\rho\mathrm{d}v^2) = -\frac{1}{1-v^2}\mathrm{d}t^2 + \frac{1}{(1-v^2)^2}\mathrm{d}v^2)$$

Note that $\epsilon = \frac{1}{1-v^2}\dot{t}$. For affinely parametrised geodesics we have

$$-\kappa = -\frac{1}{1-v^2}\dot{t}^2 + \frac{1}{(1-v^2)^2}\dot{v}^2 = -(1-v^2)\epsilon^2 + \frac{1}{(1-v^2)^2}\dot{v}^2$$

where κ is constant (affine parameter), $\kappa = 0$ for null geodesics and $\kappa > 0$ for timelike geodesics.
Now we have

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\dot{v}}{\dot{t}} = \epsilon^{-1}\frac{1}{1-v^2}\dot{v},$$

so

$$-\kappa = -\epsilon^2(1-v^2) + \epsilon^2\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2$$

(c) Explain why $\kappa = 1$ for a time-like geodesic parametrized by proper time and hence show that $\epsilon \geq 1$ for such a geodesic. Find the time-like geodesic with $\epsilon = 1$ and hence explain the physical meaning of the coordinate t.

For timelike geodesics parametrised by proper time, $g(X,X) = -1$ where X is tangent to the geodesic. But

$$g(X,X) = -\frac{1}{1-v^2}\dot{t}^2 + \frac{1}{(1-v^2)^2}\dot{v}^2 = -\kappa.$$

In this case,

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 + v^2 = 1 - \frac{1}{\epsilon^2},$$

and since the LHS is ≥ 0 we must have $\epsilon^2 \geq 1$. Also, $\epsilon \geq 0$ since t increases to the future, so in fact $\epsilon \geq 1$.

If $\epsilon = 1$,

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 + v^2 = 0.$$

Hence $v = 0$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$.
In view of $\epsilon = 1$, we find that $\dot{t} = 1$, i.e. the coordinate t agrees (up to an additive constant) with the proper time τ along a stationary geodesic through $\rho = 0$.

(d) Consider geodesics starting at the origin $\rho = 0$ with $\dot{\rho} > 0$. Sketch the trajectories of both null and time-like geodesics in the (v, t)-plane. Show that a null geodesic will reach $\rho = \infty$ at an infinite value of the affine parameter, whereas a time-like geodesic will return to $\rho = 0$ after proper time π.
[You may use without proof the definite integral

$$\int_0^\pi \frac{\mathrm{d}t}{1 - \gamma^2 \sin^2 t} = \frac{\pi}{1 - \gamma^2}$$

which is valid for $0 \leq \gamma < 1$*.]*
– skip (it's just two integrals).

2015 Question 1

Consider the Maxwell equations in vacuum,

$$\nabla^{\mu} F_{\mu\nu} = 0, \quad \nabla_{[\lambda} F_{\mu\nu]} = 0, \quad \text{with } F_{\mu\nu} = F_{\nu\mu}.$$

(a) Show that, for Minkowski spacetime, these equations imply that $F_{\mu\nu}$ satisfies a wave equation: $\partial^{\lambda}\partial_{\lambda} F_{\mu\nu} = 0$.

Working in inertial coordinates in Minkowski space, $\nabla_a = \partial_a$. Using the antisymmetry of F, the second Maxwell equation is equivalent to

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

Taking a ∂^λ derivative:

$$\begin{aligned} 0 &= \partial^\lambda \partial_\lambda F_{\mu\nu} + \partial^\lambda \partial_\mu F_{\nu\lambda} + \partial^\lambda \partial_\nu F_{\lambda\mu} \\ &= \partial^\lambda \partial_\lambda F_{\mu\nu} + \partial_\mu \partial^\lambda F_{\nu\lambda} + \partial_\nu \partial^\lambda F_{\lambda\mu} \\ &= \partial^\lambda \partial_\lambda F_{\mu\nu}. \end{aligned}$$

(b) Using local inertial coordinates, show that a $(0,2)$ tensor $T_{\mu\nu}$ satisfies

$$\nabla_\mu \nabla_\nu T_{\lambda\rho} - \nabla_\mu \nabla_\nu T_{\lambda\rho} = -R^\sigma{}_{\lambda\mu\nu} T_{\sigma\rho} - R^\sigma{}_{\rho\mu\nu} T_{\lambda\sigma}.$$

Using this result, show that the curved spacetime version of the wave equation $\partial^\lambda \partial_\lambda F_{\mu\nu} = 0$ from (a) is

$$\nabla^\lambda \nabla_\lambda F_{\mu\nu} + 2R^\lambda{}_{\mu\rho\nu} F_\lambda{}^\rho - R_\mu{}^\lambda F_{\lambda\nu} + R_\nu{}^\lambda F_{\lambda\mu} = 0.$$

With the definition of the Riemann tensor used in the course the first part of this question is trivial, but the course at the time must have defined the Riemann tensor in terms of derivatives of the Christoffel symbols.
Working in normal coordinates, where the Christoffel symbols (but *not* their derivatives) vanish, we have

$$\begin{aligned}\nabla_a \nabla_b T_{cd} - \nabla_b \nabla_a T_{cd} &= \partial_a \partial_b T_{cd} - \partial_b \partial_a T_{cd} - \partial_a \left(\Gamma^e_{bc} T_{ed} + \Gamma^e_{bd} T_{ce}\right) \\ &\quad + \partial_b \left(\Gamma^e_{ac} T_{ed} + \Gamma^e_{ad} T_{ce}\right) \\ &= (\partial_a \Gamma^e_{bc}) T_{ed} + (\partial_a \Gamma^e_{bd}) T_{ce} - (\partial_b \Gamma^e_{ac}) T_{ed} - (\partial_b \Gamma^e_{ad}) T_{ce} \\ &= \left(\partial_a \Gamma^e_{bc} - \partial_b \Gamma^e_{ac}\right) T_{ed} + \left(\partial_a \Gamma^e_{bd} - \partial_b \Gamma^e_{ad}\right) T_{ce},\end{aligned}$$

and, in normal coordinates,

$$\partial_a \Gamma^e_{bc} - \partial_b \Gamma^e_{ac} = -R^e{}_{cab}$$

Following the same ideas as in Minkowski space, we have

$$
\begin{aligned}
0 &= \nabla^\lambda \nabla_\lambda F_{\mu\nu} + \nabla^\lambda \nabla_\mu F_{\nu\lambda} + \nabla^\lambda \nabla_\nu F_{\lambda\mu} \\
&= \nabla^\lambda \nabla_\lambda F_{\mu\nu} + [\nabla^\lambda, \nabla_\mu] F_{\nu\lambda} + [\nabla^\lambda, \nabla_\nu] F_{\lambda\mu} \\
&= \nabla^\lambda \nabla_\lambda F_{\mu\nu} - R^\rho{}_\nu{}^\lambda{}_\mu F_{\rho\lambda} - R^\rho{}_\lambda{}^\lambda{}_\mu F_{\nu\rho} - R^\rho{}_\lambda{}^\lambda{}_\nu F_{\rho\mu} - R^\rho{}_\mu{}^\lambda{}_\nu F_{\lambda\rho} \\
&= \nabla^\lambda \nabla_\lambda F_{\mu\nu} - 2R^\rho{}_\nu{}^\lambda{}_\mu F_{\rho\lambda} + R_\mu{}^\rho F_{\nu\rho} + R_\nu{}^\rho F_{\rho\mu}.
\end{aligned}
$$

(c) The Maxwell action is

$$S = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right).$$

Show that this action is invariant for a conformal transformation of the metric, that is, show that $S[g_{\mu\nu}, F_{\lambda\rho}] = S[\tilde{g}_{\mu\nu}, F_{\lambda\rho}]$ for $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $\Omega(x)$ is a function.

We have $\sqrt{-\tilde{g}} = \sqrt{-\det \tilde{g}} = \Omega^4 \sqrt{-g}$. On the other hand, we have

$$-\frac{1}{4}(\tilde{g}^{-1})^{\mu\rho}(\tilde{g}^{-1})^{\nu\sigma} F_{\rho\sigma} F_{\mu\nu} = \Omega^{-4} \left(-\frac{1}{4}(g^{-1})^{\mu\rho}(g^{-1})^{\nu\sigma} F_{\rho\sigma} F_{\mu\nu} \right),$$

so the action S is invariant under $g \mapsto \tilde{g}$.

(d) Show that, in four spacetime dimensions, the Maxwell energy-momentum tensor,

$$T_{\mu\nu} = F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F^{\lambda\rho} F_{\lambda\rho},$$

is traceless, that is, $T^{\mu}{}_{\mu} = 0$. Considering the definition of $T_{\mu\nu}$ in terms of the action, relate this result to the invariance of the action for conformal transformations, from (c).

We have

$$T^{\mu}{}_{\mu} = F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}g^{\mu}{}_{\mu}F^{\lambda\rho}F_{\lambda\rho} = 0,$$

since we are in four dimensions.
The energy-momentum tensor is defined in terms of the Lagrangian density $\mathcal{L}$ (where $S = \int \mathrm{d}^4x\mathcal{L}$) by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{(\delta g^{-1})^{\mu\nu}},$$

and so, varying the inverse metric

$$\delta S = \int \mathrm{d}^4x\delta\mathcal{L} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,T_{\mu\nu}(\delta g^{-1})^{\mu\nu}$$

Now consider an infinitesimal conformal transformation, $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$ where $\delta g_{\mu\nu} = \omega g_{\mu\nu}$. In this case, $\delta g^{\mu\nu} = -\omega g^{\mu\nu}$, and we see that the action varies by

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\omega T^{\mu}{}_{\mu},$$

but, since the action is invariant under conformal transformations, and since ω is arbitrary, we must have $T^{\mu}{}_{\mu} = 0$.

2015 Question 2

There is a class of metrics which admit coordinates such that

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu,$$

with k_μ satisfying $\eta^{\mu\nu} k_\mu k_\nu = 0$, where
$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric.
(a) Look for $g^{\mu\nu}$ of the form $g^{\mu\nu} = \eta^{\mu\nu} + \theta k^\mu k^\nu$, and then show that the covector k_μ is null with respect to the metric $g_{\mu\nu}$.

If we define $k^\mu = \eta^{\mu\nu} k_\nu$, then the inverse metric should satisfy $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$, i.e.

$$\delta^\mu_\rho = \left(\eta^{\mu\nu} + \theta k^\mu k^\nu\right)\left(\eta_{\nu\rho} + \phi k_\nu k_\rho\right)$$

$$= \delta^\mu_\rho + \left(\phi + \theta + \phi\theta k^\nu k_\nu\right) k^\mu k_\rho$$

$$\Rightarrow \theta = -\phi.$$

Hence

$$g^{\mu\nu} k_\mu k_\nu = (\eta^{\mu\nu} - \phi k^\mu k^\nu) k_\mu k_\nu = 0.$$

Note that this means we can raise or lower indices on k using either g or η.

(b) Show that $\Gamma^{\lambda}_{\mu\nu}k^{\mu}k^{\nu} = 0$ and $\Gamma^{\lambda}_{\mu\nu}k_{\lambda}k^{\mu} = 0$. Use this to show that if k_{μ} is geodesic with respect to the Minkowski metric, $k_{\mu}\eta^{\mu\lambda}\partial_{\lambda}k_{\nu} = 0$, then it is also geodesic with respect to the curved metric $g_{\mu\nu}$, $k^{\mu}\nabla_{\mu}k_{\nu} = 0$.

We have

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)$$

$$= \frac{1}{2}(\eta^{\lambda\rho} - \phi k^\lambda k^\rho) \left(\partial_\mu(\phi k_\nu k_\rho) + \partial_\nu(\phi k_\mu k_\rho) - \partial_\rho(\phi k_\mu k_\nu)\right),$$

so

$$\Gamma^\lambda_{\mu\nu} k^\mu k^\nu = \frac{1}{2}(\eta^{\lambda\rho} - \phi k^\lambda k^\rho) \left(\partial_\mu(\phi k_\nu k_\rho) + \partial_\nu(\phi k_\mu k_\rho) - \partial_\rho(\phi k_\mu k_\nu)\right) k^\mu k^\nu$$

Using the fact that $k^\mu k_\mu = 0$ and that we can raise/lower using η (and $\partial\eta = 0$), we find

$$\left(\partial_\mu(\phi k_\nu k_\rho) + \partial_\nu(\phi k_\mu k_\rho) - \partial_\rho(\phi k_\mu k_\nu)\right) k^\mu k^\nu = \phi k^\mu (\partial_\mu k_\nu) k^\nu k_\rho + \phi k^\nu (\partial_\nu k_\mu) k^\mu k_\rho$$

$$= \phi k^\mu \partial_\mu \left(k_\nu k^\nu\right) k_\rho = 0$$

Similarly we have

$$\Gamma^{\lambda}_{\mu\nu} k_\lambda k^\mu = \frac{1}{2} k^\rho k^\mu \left(\partial_\mu(\phi k_\nu k_\rho) + \partial_\nu(\phi k_\mu k_\rho) - \partial_\rho(\phi k_\mu k_\nu) \right)$$

$$= \frac{1}{2} \phi k_\nu k^\mu (\partial_\mu k_\rho) k^\rho - \frac{1}{2} \phi k_\nu k^\rho (\partial_\rho k_\mu) k^\mu = 0$$

Next we calculate

$$k^\mu \nabla_\mu k_\nu = k^\mu \partial_\mu k_\nu - k^\mu \Gamma^{\lambda}_{\mu\nu} k_\lambda = k^\mu \partial_\mu k_\nu,$$

so if k is geodesic w.r.t. η then it is geodesic w.r.t. g.

(c) Consider the metric $g_{\mu\nu}$ in this class for which

$$\phi = \frac{2M}{r}, \qquad k_\mu = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right),$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Using the result in (b), show that k_μ is geodesic. Show also that $k_\mu \mathrm{d}x^\mu = \mathrm{d}\tilde{t} + \mathrm{d}r$, where $x^0 = \tilde{t}$. Finally, show that the spacetime in question is the Schwarzschild spacetime, by putting the metric in the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right).$$

[Hint: Look for a coordinate change of the type $\tilde{t} = t + h(r)$.]

k_μ is geodesic in Minkowski space, as it is tangent to the family of geodesics $r = t$, at fixed angular coordinates. Hence, by (b), it is geodesic in the curved space.

$$k_\mu \mathrm{d}x^\mu = \mathrm{d}\tilde{t} + \frac{x}{r}\mathrm{d}x + \frac{y}{r}\mathrm{d}y + \frac{z}{r}\mathrm{d}z = \mathrm{d}\tilde{t} + \mathrm{d}r$$

Now, the metric is

$$g = -\mathrm{d}\tilde{t}^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \phi\left(\mathrm{d}\tilde{t} + \mathrm{d}r\right)^2$$

$$= -\left(1 - \frac{2M}{r}\right)\mathrm{d}\tilde{t}^2 + \frac{4M}{r}\mathrm{d}\tilde{t}\mathrm{d}r + \left(1 + \frac{2M}{r}\right)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right),$$

transitioning standard spherical polars. Now setting $\tilde{t} = t + h(r)$, we find that the metric is

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(\frac{4M}{r} - 2h'\left(1 - \frac{2M}{r}\right)\right)\mathrm{d}t\mathrm{d}r$$
$$+ \left(\left(1 + \frac{2M}{r}\right)(1 - (h')^2) + \frac{4M}{r}h'\right)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right).$$

$$
\begin{aligned}
\mathrm{d}s^2 = &- \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(\frac{4M}{r} - 2h' \left(1 - \frac{2M}{r}\right)\right) \mathrm{d}t\mathrm{d}r \\
&+ \left(1 + \frac{2M}{r} - \left(1 - \frac{2M}{r}\right) (h')^2 + \frac{4M}{r} h'\right) \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right).
\end{aligned}
$$

We can eliminate the cross term by setting $h' = \frac{2M}{r}\left(1 - \frac{2M}{r}\right)^{-1} = \frac{2M}{r-2M}$, i.e. $h = 2M\log(r - 2M) + C$. In this case the coefficient of $\mathrm{d}r^2$ is

$$
\begin{aligned}
1 + \frac{2M}{r} - \frac{4M^2}{r(r-2M)} + \frac{8M^2}{r(r-2M)} &= \frac{r^2 - 2Mr + 2Mr - 4M^2 + 4M^2}{r(r-2M)} \\
&= \frac{r}{r-2M} = \left(1 - \frac{2M}{r}\right)^{-1}
\end{aligned}
$$

So the metric is the Schwarzschild metric.

Showing an object is a tensor

Without assuming the connection is torsion-free, show that

$$T(X, Y, \mu) := \mu\left(\nabla_X Y - \nabla_Y X - [X, Y]\right)$$

is a tensor, where X and Y are vector fields and μ is a covector field.

Need to show linearity in all three arguments. Linearity in third argument is obvious (covectors form a vector space at each point). Linearity in second argument will follow from linearity in the first argument together with antisymmetry:

$$T(X, Y, \mu) = -T(Y, X, \mu).$$

Now we expand (for a vector field X' and scalar field a)

$$
\begin{aligned}
T(aX + X', Y, \mu) &= \mu\left(\nabla_{aX+X'} Y - \nabla_Y (aX + X') - [aX + X', Y]\right) \\[2ex]
&= \mu\big(a\nabla_X Y + \nabla_{X'} Y - Y(a)X - a\nabla_Y X \\
&\qquad - \nabla_Y X' - a[X, Y] + Y(a)X - [X', Y]\big) \\[2ex]
&= a\mu\left(\nabla_X Y - \nabla_Y X - [X, Y]\right) \\
&\quad + \mu\left(\nabla_{X'} Y - \nabla_Y X' - [X', Y]\right) \\[2ex]
&= aT(X, Y, \mu) + T(X', Y, \mu).
\end{aligned}
$$