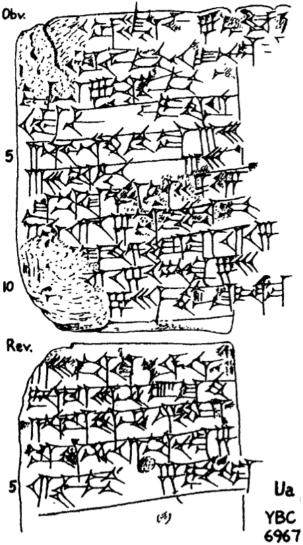
O1 History of Mathematics
Lecture IX
Classical algebra: equation solving
1800BC – AD1800

Monday 5th November 2018
(Week 5)

Summary

- Early quadratic equations
- Cubic and quartic equations
- Further 16th-century developments
- 17th century ideas
- 18th century ideas
- Looking back

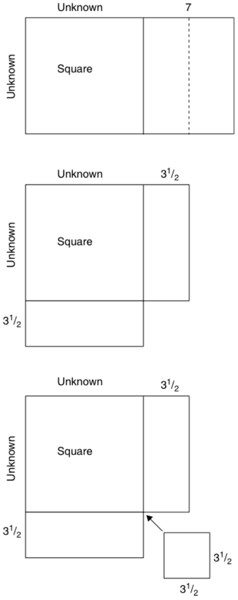
Completing the square, c. 1800 BC



A Babylonian scribe, clay tablet BM 13901, c. 1800 BC:

A reciprocal exceeds its reciprocal by 7. What are the reciprocal and its reciprocal? Break in half the 7 by which the reciprocal exceeds its reciprocal, and $3\frac{1}{2}$ will come up. Multiply $3\frac{1}{2}$ by $3\frac{1}{2}$ and $12\frac{1}{4}$ will come up. Append 60, the area, to the $12\frac{1}{4}$ which came up for you and $72\frac{1}{4}$ will come up. What is the square-side of $72\frac{1}{4}$? $8\frac{1}{2}$. Put down $8\frac{1}{2}$ and $8\frac{1}{2}$ and subtract $3\frac{1}{2}$ from one of them; append $3\frac{1}{2}$ to one of them. One is 12, the other is 5. The reciprocal is 12, its reciprocal 5.

Completing the square, c. 1800 BC



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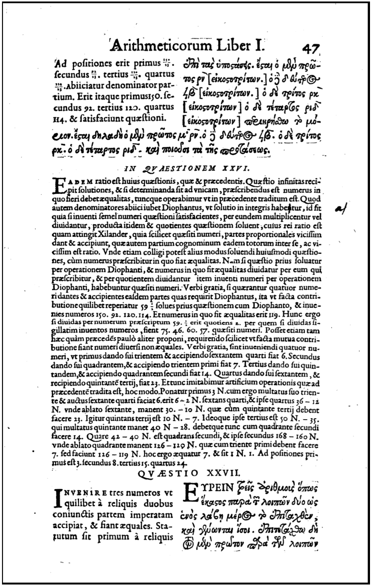
Points to note

- We have used the word ‘equation’ without writing down anything in symbols
- Solution recipe derived from geometrical insight
- Not (explicitly) a general solution — but reader ought to be able to adapt the method
- Is this algebra? Geometrical algebra?

Diophantus of Alexandria (3rd century AD)



Ad positiones erit primus $\frac{150}{23}$. secundus $\frac{92}{23}$. tertius $\frac{120}{23}$. quartus $\frac{114}{23}$. Abiiciatur denominator partium. Erit itaque primus 150. secundus 92. tertius 120. quartus 114. & satisfaciunt quæstioni.

ἐπὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν πρῶτος ρν [εἰκοστοτρίτων.] ὁ δὲ δεύτερος ϟβ [εἰκοστοτρίτων.] ὁ δὲ τρίτος ρκ [εἰκοστοτρίτων] ὁ δὲ τέταρτος ριδ [εἰκοστοτρίτων] περιῃρήσθω τὸ μόριον. ἔσται δηλαδὴ ὁ μὲν πρῶτος μο ρν. ὁ δὲ δεύτερος ϟβ. ὁ δὲ τρίτος ρκ. ὁ δὲ τέταρτος ριδ. καὶ ποιοῦσι τὰ τῆς προτάσεως.

IN QVAESTIONEM XXVI.

EADEM ratio est huius quæstionis, quæ & præcedentis. Quæstio infinitas recipit solutiones, & si determinanda sit ad vnicam, præscribendus est numerus in quo fieri debet æqualitas, tuncque operabimur vt in præcedente traditum est. Quod autem denominatores abiici iubet Diophantus, vt solutio in integris habeatur, id fit quia si inuenti semel numeri quæstioni satisfacientes, per eundem multiplicentur vel diuidantur, producta itidem & quotientes quæstionem soluent, cuius rei ratio est quam attingit Xilander, quia scilicet quæsiti numeri, partes proportionales vicissim dant & accipiunt, quæ autem partium cognominum eadem totorum inter se, ac vicissim est ratio. Vnde etiam colligi potest alius modus soluendi huiusmodi quæstiones, cùm numerus præscribitur in quo fiat æqualitas. Nam si quæstio prius soluatur per operationem Diophanti, & numerus in quo fit æqualitas diuidatur per eum qui præscribitur, & per quotientem diuidantur item inuenti numeri per operationem Diophanti, habebuntur quæsiti numeri. Verbi gratia, si quærantur quatuor numeri dantes & accipientes easdem partes quas requirit Diophantus, ita vt facta contributione quilibet reperiatur 59 $\frac{1}{2}$ solues prius quæstionem cum Diophanto, & inuenies numeros 150. 92. 120. 114. Et numerus in quo fit æqualitas erit 119. Hunc ergo si diuidas per numerum præscriptum 59. $\frac{1}{2}$ erit quotiens 2. per quem si diuidas sigillatim inuentos numeros, fient 75. 46. 60. 57. quæsiti numeri. Posset etiam tam hæc quàm præcedẽs paulò aliter proponi, requirendo scilicet vt facta mutua contributione fiant numeri diuersi non æquales. Verbi gratia, sint inueniendi quatuor numeri, vt primus dando sui trientem & accipiendo sextantem quarti fiat 6. Secundus dando sui quadrantem, & accipiendo trientem primi fiat 7. Tertius dando sui quintandem, & accipiendo quadrantem secundi fiat 14. Quartus dando sui sextantem, & recipiendo quintantẽ tertij, fiat 23. Et tunc imitabimur artificium operationis quæ ad præcedentẽ tradita est, hoc modo. Ponatur primus 3 N. cum ergo multatus suo triente & auctus sextante quarti faciat 6. erit 6 − 2 N. sextans quarti, & ipse quartus 36 − 12 N. vnde ablato sextante, manent 30. − 10 N. quæ cùm quintante tertij debent facere 23. Igitur quintans tertij est 10 N. − 7. Ideoque ipse tertius est 50 N. − 35. qui multatus quintante manet 40 N − 28. debetque tunc cum quadrante secundi facere 14. Quare 42 − 40 N. est quadrans secundi, & ipse secundus 168 − 160 N. vnde ablato quadrante manent 126 − 120 N. quæ cum triente primi debent facere 7. sed faciunt 126 − 119 N. hoc ergo æquatur 7. & fit 1 N. 1. Ad positiones primus est 3. secundus 8. tertius 15. quartus 24.

QVÆSTIO XXVII.

INVENIRE tres numeros vt quilibet à reliquis duobus coniunctis partem imperatam accipiat, & fiant æquales. Statutum sit primum à reliquis

ΕΥΡΕΙΝ τρεῖς ἀριθμοὺς ὅπως ἕκαστος παρὰ τῶν λοιπῶν δύο ὡς ἑνὸς λάβῃ μέρος τὸ ἐπιταχθὲν, καὶ γίνωνται ἴσοι. ἐπιτετάχθω δὴ τὸν μὲν πρῶτον παρὰ τῶν λοιπῶν

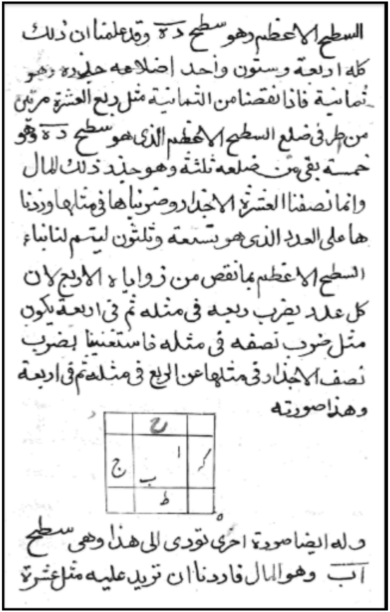
Problem I.27: Find two numbers such that their sum and product are given numbers

Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

Noted six cases of equations:

1. Squares are equal to roots ($ax^2 = bx$)
2. Squares are equal to numbers ($ax^2 = c$)
3. Roots are equal to numbers ($bx = c$)
4. Squares and roots are equal to numbers ($ax^2 + bx = c$)
5. Squares and numbers are equal to roots ($ax^2 + c = bx$)
6. Roots and numbers are equal to squares ($bx + c = ax^2$)

Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

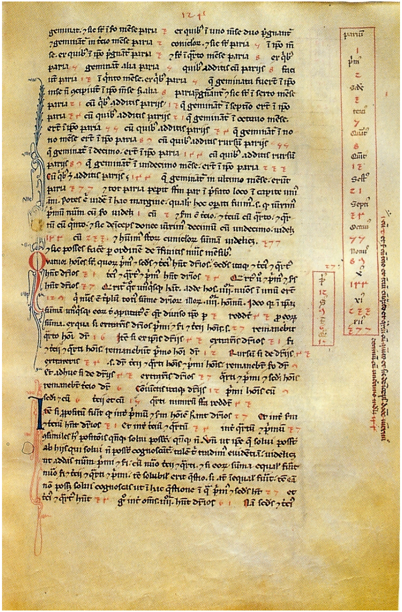

السطح الاعظم وهو سطح د ه وقد علمنا ان ذلك
كله اربعة وستون واحد اضلاعه جذره وهو
ثمانية فاذا نقصنا من الثمانية مثل ربع العشرة مرتين
من طرفي ضلع السطح الاعظم الذي هو سطح د ه وهو
خمسة بقي من ضلعه ثلثة وهو جذر ذلك المال
وانما نصفنا العشرة الاجذار وضربناها في مثلها وزدنا
ها على العدد الذي هو تسعة وثلثون ليتم لنا بناء
السطح الاعظم بما نقص من زوايا ه الاربع لان
كل عدد يضرب ربعه في مثله ثم في اربعة يكون
مثل ضرب نصفه في مثله فاستغنينا بضرب
نصف الاجذار في مثلها عن الربع في مثله ثم في اربعة
وهذا صورته
وله ايضا صورة اخرى تودي الى هذا وهي سطح
آب وهو المال فاردنا ان نزيد عليه مثل عشرة

An algorithm for case (4) on the previous slide

Leonardo of Pisa (Fibonacci) (c. 1175–c. 1240/50)

Liber abaci (or *Liber abbaci*), Pisa, 1202:

- included al-Khwārizmi's recipes
- geometrical demonstrations and lots of examples
- didn't go very far beyond predecessors, but began transmission of Islamic ideas to Europe



Cubic equations (1)

Italy, early 16th century:

solutions to cubics of the form $x^3 + px = q$

- found by Scipione del Ferreo (or Ferro) (c. 1520)
- taught to Antonio Maria Fiore (pupil)
- and Annibale della Nave (son-in-law)
- rediscovered by Niccolò Tartaglia (1535)
- passed in rhyme to Girolamo Cardano (1539)

Cubic equations (2)

$$x^3 + px = q$$

When the cube with the things next after
Together equal some number apart
Find two others that by this differ
And this you will keep as a rule
That their product will always be equal
To a third cubed of the number of things
The difference then in general between
The sides of the cubes subtracted well
Will be your principal thing.

(Tartaglia, 1546; see: *Mathematics emerging*, §12.1.1)

Cubic equations (3)

$$x^3 + px = q$$

When the cube with the things next after
Together equal some number apart
Find two others that by this differ
And this you will keep as a rule
That their product will always be equal
To a third cubed of the number of things
The difference then in general between
The sides of the cubes subtracted well
Will be your principal thing.

Interpretation of Tartaglia's rhyme:

Find u, v such that

$$u - v = q, \quad uv = \left(\frac{p}{3}\right)^3.$$

Then

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

NB: In an equation $y^3 + ay^2 + by + c = 0$ we can put $y = x - \frac{a}{3}$ to remove the square term, so this solution is general.

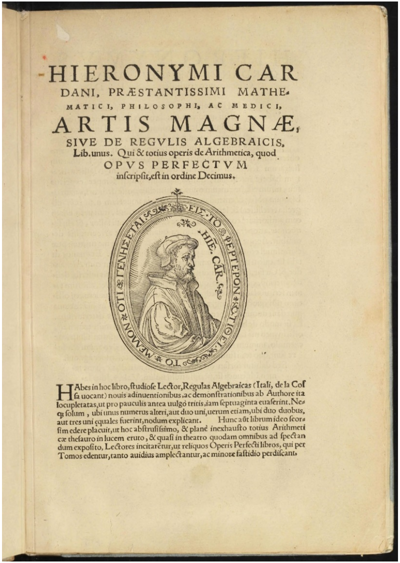
Cubic equations (4)

In modern terms, one of the solutions of the equation $ax^3 + bx^2 + cx + d = 0$ has the form

$$x = \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

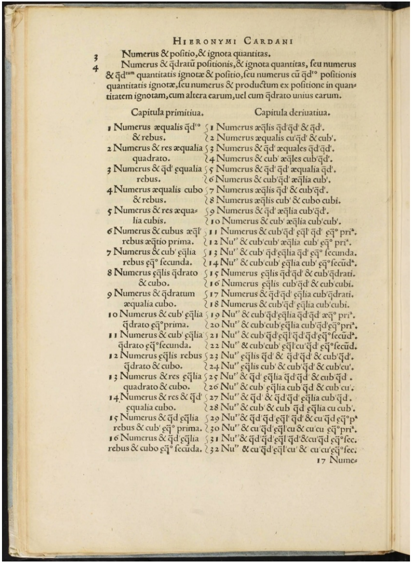
with similar expressions (in radicals) for the remaining two roots

Cardano's *Ars magna, sive de regulis algebraicis* (1545)



HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE-
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.



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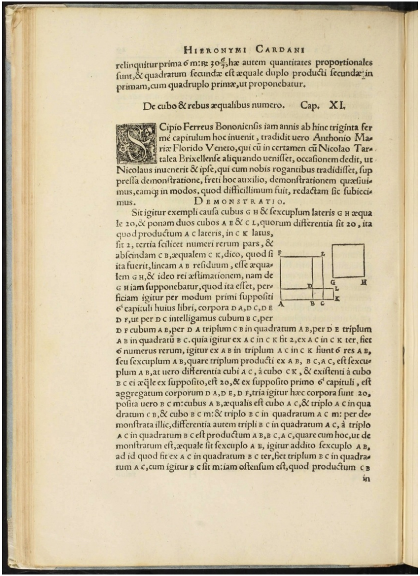
3 Numerus & positio, & ignota quantitas.
4 Numerus & q̄dratũ positionis, & ignota quantitas, seu numerus & q̄d^{tum} quantitatis ignotæ & positio, seu numerus cũ q̄d^{to} positionis quantitatis ignotæ, seu numerus & productum ex positione in quantitatem ignotam, cum altera earum, uel cum q̄drato unius earum.

Capitula primitiua.	Capitula deriuatiua.
1 Numerus æqualis q̄d^{to} & rebus.	1 Numerus æq̄lis q̄d' q̄d' & q̄d'. 2 Numerus æqualis cu' q̄d' & cub'.
2 Numerus & res æqualia quadrato.	3 Numerus & q̄d' æquales q̄d' q̄d'. 4 Numerus & cub' æq̄les cub' q̄d'.
3 Numerus & q̄d' ęqualia rebus.	5 Numerus & q̄d' q̄d' æqualia q̄d'. 6 Numerus & cub' q̄d' æq̄lia cub'.
4 Numerus æqualis cubo & rebus.	7 Numerus æq̄lis q̄d' & cub' q̄d'. 8 Numerus æq̄lis cub' & cubo cubi.
5 Numerus & res æqualia cubis.	9 Numerus & q̄d' æq̄lia cub' q̄d'. 10 Numerus & cub' æq̄lia cub' cub'.
6 Numerus & cubus æq̄l' rebus æq̄tio prima.	11 Numerus & cub' q̄d' ęq̄l' q̄d' ęq̄° pri^a. 12 Nu^{r}' & cub' cub' æq̄lia cub' ęq̄° pri^a.
7 Numerus & cub' ęq̄lia rebus ęq̄° secunda.	13 Nu^{r}' & cub' q̄d' ęq̄lia q̄d' ęq̄° secunda. 14 Nu^{r}' & cub' cub' ęq̄lia cub' ęq̄° secũd^a.
8 Numerus ęq̄lis q̄drato & cubo.	15 Numerus ęq̄lis q̄d' q̄d' & cub' q̄drati. 16 Numerus ęq̄lis cub' q̄d' & cub' cubi.
9 Numerus & q̄dratum æqualia cubo.	17 Numerus & q̄d' q̄d' ęq̄lia cub' q̄drati. 18 Numerus & cub' q̄d' ęq̄lia cub' cubi.
10 Numerus & cub' ęq̄lia q̄drato ęq̄° prima.	19 Nu^{r}' & cub' q̄d' ęq̄lia q̄d' q̄d' æq̄° pri^a. 20 Nu^{r}' & cub' cub' ęq̄lia cub' q̄d' ęq̄° pri^a.
11 Numerus & cub' ęq̄lia q̄drato ęq̄° secunda.	21 Nu^{r}' & cub' q̄d' ęq̄l' q̄d' q̄d' ęq̄° secũd^a. 22 Nu^{r}' & cub' cub' ęq̄l' cu' q̄d' ęq̄° secũd.
12 Numerus ęq̄lis rebus q̄drato & cubo.	23 Nu^{r}' ęq̄lis q̄d' & q̄d' q̄d' & cub' q̄d'. 24 Nu^{r}' ęq̄lis cub' & cub' q̄d' & cub' cu'.
13 Numerus & res ęq̄lia quadrato & cubo.	25 Nu^{r}' & q̄d' ęq̄lia q̄d' q̄d' & cub' q̄d'. 26 Nu^{r}' & cub' ęq̄lia cub' q̄d & cub' cu'.
14 Numerus & res & q̄d' ęqualia cubo.	27 Nu^{r}' & q̄d' & q̄d' q̄d' ęq̄lia cub' q̄d'. 28 Nu^{r}' & cub' & cub' q̄d' ęq̄lia cu cub'.
15 Numerus & q̄d' ęq̄lia rebus & cub' ęq̄° prima.	29 Nu^{r}' & q̄d' q̄d' ęq̄l' q̄d' & cu' q̄d ęq̄° p^a. 30 Nu^{r}' & cu' q̄d' ęq̄l' cu & cu' cu ęq̄° pri^a.
16 Numerus & q̄d' ęq̄lia rebus & cubo ęq̄° secũda.	31 Nu^{r}' & q̄d' q̄d' ęq̄l' q̄d' & cu' q̄d' ęq̄° sec. 32 Nu^{r}' & cu' q̄d' ęq̄l' cu' & cu' cu' ęq̄° sec.

17 Nume-

Cardano on the cubic



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relinquitur prima 6 m: ℞ 30 6/7, hæ autem quantitates proportionales ſunt, & quadratum ſecundæ eſt æquale duplo producti ſecundæ in primam, cum quadruplo primæ, ut proponebatur.

De cubo & rebus æqualibus numero. Cap. XI.

SCipio Ferreus Bononienſis iam annis ab hinc triginta ferme capitulum hoc inuenit, tradidit uero Anthonio Mariæ Florido Veneto, qui cũ in certamen cũ Nicolao Tartalea Brixellenſe aliquando ueniſſet, occaſionem dedit, ut Nicolaus inuenerit & ipſe, qui cum nobis rogantibus tradidiſſet, ſuppreſſa demonſtratione, freti hoc auxilio, demonſtrationem quæſiuimus, eamq; in modos, quod difficillimum fuit, redactam ſic ſubiecimus.

DEMONSTRATIO.

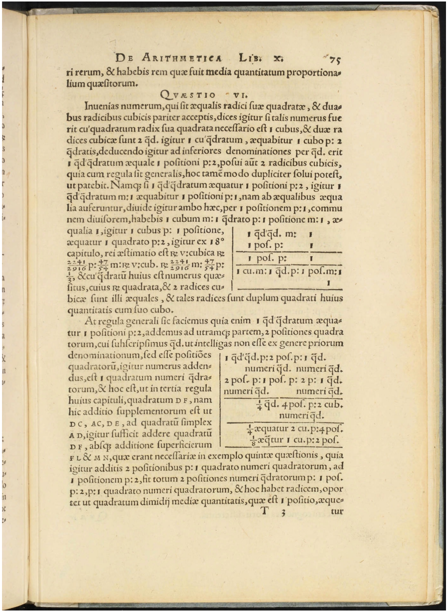
Sit igitur exempli cauſa cubus G H & ſexcuplum lateris G H æquale 20, & ponam duos cubos A E & C L, quorum differentia ſit 20, ita quod productum A C lateris, in C K latus, ſit 2, tertia ſcilicet numeri rerum pars, & abſcindam C B, æqualem C K, dico, quod ſi ita fuerit, lineam A B reſiduum, eſſe æqualem G H, & ideo rei æſtimationem, nam de G H iam ſupponebatur, quod ita eſſet, perficiam igitur per modum primi ſuppoſiti 6ⁱ capituli huius libri, corpora D A, D C, D E D F, ut per D C intelligamus cubum B C, per D F cubum A B, per D A triplum C B in quadratum A B, per D E triplum A B in quadratũ B C. quia igitur ex A C in C K fit 2, ex A C in C K ter, fiet 6 numerus rerum, igitur ex A B in triplum A C in C K fiunt 6 res A B, ſeu ſexcuplum A B, quare triplum producti ex A B, B C, A C, eſt ſexcuplum A B, at uero differentia cubi A C, à cubo C K, & exiſtenti à cubo B C ei æq̃le ex ſuppoſito, eſt 20, & ex ſuppoſito primo 6ⁱ capituli, eſt aggregatum corporum D A, D E, D F, tria igitur hæc corpora ſunt 20, poſita uero B C m: cubus A B, æqualis eſt cubo A C, & triplo A C in quadratum C B, & cubo B C m: & triplo B C in quadratum A C m: per demonſtrata illic, differentia autem tripli B C in quadratum A C, à triplo A C in quadratum B C eſt productum A B, B C, A C, quare cum hoc, ut demonſtratum eſt, æquale ſit ſexcuplo A B, igitur addito ſexcuplo A B, ad id quod fit ex A C in quadratum B C ter, fiet triplum B C in quadratum A C, cum igitur B C ſit m: iam oſtenſum eſt, quod productum C B

in

- Geometrical justification remains
- General solution (to particular case), rather than example to be followed
- Make substitution $x = y - \frac{a}{3}$ in $y^3 + ax^2 + bx + c = d$ to suppress square term and obtain equation of the form $x^3 + px = q$ — manipulation of equations prior to solution

Quartic equations (1)

General solution discovered (again on a case-by-case basis) by Lodovico Ferrari (c. 1540) and published by Cardano, in the form of worked examples, alongside solution of cubic



ri rerum, & habebis rem quæ fuit media quantitatum proportionalium quæsitorum.

QVÆSTIO VI.

Inuenias numerum, qui sit æqualis radici suæ quadratæ, & duabus radicibus cubicis pariter acceptis, dices igitur si talis numerus fuerit cu'quadratum radix sua quadrata necessario est 1 cubus, & duæ radices cubicæ sunt 2 q̄d. igitur 1 cu'q̄dratum, æquabitur 1 cubo p: 2 q̄dratis, deducendo igitur ad inferiores denominationes per q̄d. erit 1 q̄d'q̄dratum æquale 1 positioni p:2, posui aūt 2 radicibus cubicis, quia cum regula sit generalis, hoc tamē modo dupliciter solui potest, ut patebit. Namq; si 1 q̄d'q̄dratum æquatur 1 positioni p:2, igitur 1 q̄d'q̄dratum m:1 æquabitur 1 positioni p:1, nam ab æqualibus æqualia auferuntur, diuide igitur ambo hæc, per 1 positionem p:1, communem diuisorem, habebis 1 cubum m:1 q̄drato p:1 positione m:1, æqualia 1, igitur 1 cubus p: 1 positione, æquatur 1 quadrato p:2, igitur ex 18° capitulo, rei æstimatio est R v:cubica R $\frac{2241}{2916}$ p: $\frac{47}{54}$ m:R v:cub. R $\frac{2241}{2916}$ m: $\frac{47}{54}$ p: $\frac{1}{3}$, & cu'q̄dratū huius est numerus quæsitus, cuius R quadrata, & 2 radices cubicæ sunt illi æquales, & tales radices sunt duplum quadrati huius quantitatis cum suo cubo.

1 q̄d'q̄d. m: 1
1 pos. p: 1
1 pos. p: 1
1 cu. m: 1 q̄d. p: 1 pos. m: 1
1

At regula generali sic faciemus quia enim 1 q̄d'q̄dratum æquatur 1 positioni p:2, addemus ad utramq; partem, 2 positiones quadratorum, cui subscripsimus q̄d. ut intelligas non esse ex genere priorum denominationum, sed esse positiōes quadratorū, igitur numerus addendus, est 1 quadratum numeri q̄dratorum, & hoc est, ut in tertia regula huius capituli, quadratum DF, nam hic additio supplementorum est ut DC, AC, DE, ad quadratū simplex AD, igitur sufficit addere quadratū DF, absq; additione superficierum FL & MN, quæ erant necessariæ in exemplo quintæ quæstionis, quia igitur additis 2 positionibus p: 1 quadrato numeri quadratorum, ad 1 positionem p: 2, fit totum 2 positiones numeri q̄dratorum p: 1 pos. p: 2, p: 1 quadrato numeri quadratorum, & hoc habet radicem, oportet ut quadratum dimidij mediæ quantitatis, quæ est 1 positio, æque-

1 q̄d'q̄d. p: 2 pos. p: 1 q̄d.
numeri q̄d. numeri q̄d.
2 pos. p: 1 pos. p: 2 p: 1 q̄d.
numeri q̄d. numeri q̄d.
$\frac{1}{4}$ q̄d. 4 pos. p: 2 cub.
numeri q̄d.
$\frac{1}{4}$ æquatur 2 cu. p: 4 pos.
$\frac{1}{8}$ æq̄tur 1 cu. p: 2 pos.

T 3 tur

Quartic equations (2)

In modern terms, suppose that

$$x^4 = px^2 + qx + r.$$

Add $2yx^2 + y^2$ to each side to give

$$(x^2 + y)^2 = (p + 2y)x^2 + qx + (r + y^2).$$

Now we seek y such that the right hand side is a perfect square:

$$8y^3 + 4py^2 + 8ry + (4pr - q^2) = 0.$$

So the problem is reduced to solving a cubic equation and then a quadratic.

NB: In an equation $y^4 + ay^3 + by^2 + cy + d = 0$ we can put $y = x - \frac{a}{4}$ to remove the cube term, so this solution is general.

Quartic equations (3)

Formulae for the solutions of the general quartic equation, in all their unedifying glory, may be found at:

http://planetmath.org/QuarticFormula

Cardano's *Ars Magna* may also be found online here

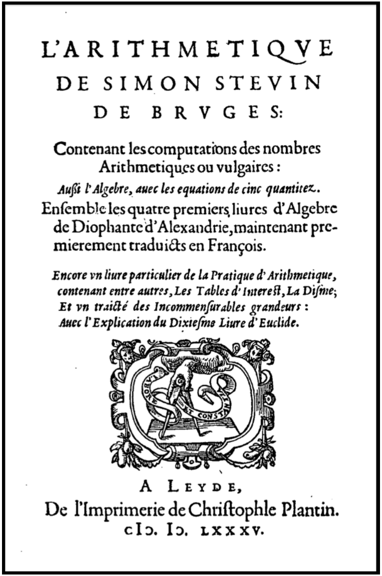
Further 16th-century developments

Rafael Bombelli, *L'algebra* (1572):

- heavily influenced by Cardano
- equation solving, new notation
- exploration of complex numbers

  [to be dealt with in a later lecture]

Further 16th-century developments


L'ARITHMETIQVE
DE SIMON STEVIN
DE BRVGES:
Contenant les computations des nombres Arithmetiques ou vulgaires:
Aussi l'Algebre, auec les equations de cinc quantitez.
Ensemble les quatre premiers liures d'Algebre de Diophante d'Alexandrie, maintenant premierement traduicts en François.
Encore vn liure particulier de la Pratique d'Arithmetique, contenant entre autres, Les Tables d'Interest, La Disme; Et vn traicté des Incommensurables grandeurs: Auec l'Explication du Dixiesme Liure d'Euclide.
A LEYDE,
De l'Imprimerie de Christophle Plantin.
cIↃ. IↃ. LXXXV.

Simon Stevin, *L'arithmetique … aussi l'algebre* (1585):

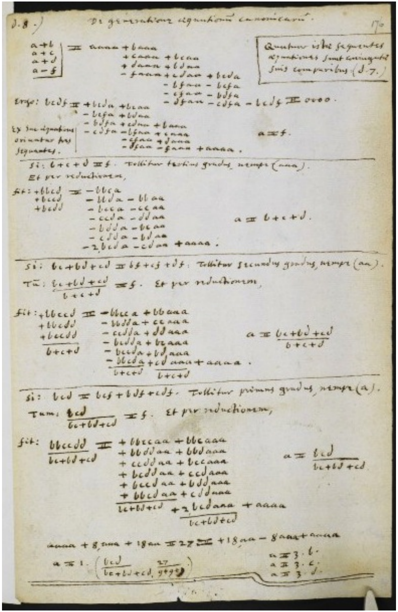
- heavily influenced by Cardano through Bombelli
- appended his treatise on decimal notation

Further 16th-century developments

François Viète (1590s):

- links between algebra and geometry
- (algebra as 'analysis' or 'analytic art')
- notation [recall Lecture III]
- numerical methods for solving equations

Thomas Harriot (c. 1600)



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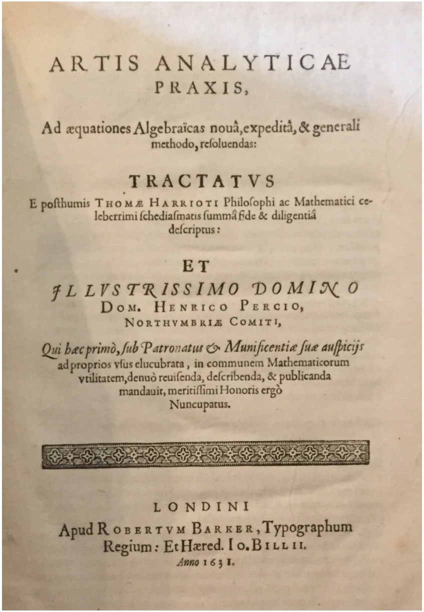
Note:

- notation [see lecture III];
- appearance of polynomials as products of linear factors.

Thomas Harriot (1631)



ARTIS ANALYTICAE
PRAXIS,

Ad æquationes Algebraïcas nouâ, expeditâ, & generali
methodo, resoluendas:

TRACTATVS

E posthumis THOMÆ HARRIOTI Philosophi ac Mathematici celeberrimi schediasmatis summâ fide & diligentiâ
descriptus:

ET

ILLVSTRISSIMO DOMINO
DOM. HENRICO PERCIO,
NORTHVMBRIÆ COMITI,

Qui hæc primò, sub Patronatus & Munificentiæ suæ auspicijs
ad proprios vsus elucubrata, in communem Mathematicorum
vtilitatem, denuò reuisenda, describenda, & publicanda
mandauit, meritissimi Honoris ergò
Nuncupatus.

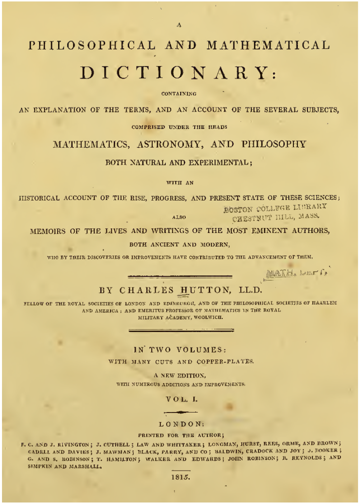
LONDINI
Apud ROBERTVM BARKER, Typographum
Regium: Et Hæred. Io. BILLII.
Anno 1631.

Some of Harriot's ideas found their way into his *Artis analyticae praxis* (*The practice of the analytic art*), published posthumously in 1631

But editors did not permit negative or imaginary roots [to be discussed further in a later lecture]

See *Mathematics emerging*, §12.2.1.

Commentary on Harriot


A

PHILOSOPHICAL AND MATHEMATICAL

DICTIONARY:

CONTAINING

AN EXPLANATION OF THE TERMS, AND AN ACCOUNT OF THE SEVERAL SUBJECTS,

COMPRISED UNDER THE HEADS

MATHEMATICS, ASTRONOMY, AND PHILOSOPHY

BOTH NATURAL AND EXPERIMENTAL;

WITH AN

HISTORICAL ACCOUNT OF THE RISE, PROGRESS, AND PRESENT STATE OF THESE SCIENCES;

ALSO

MEMOIRS OF THE LIVES AND WRITINGS OF THE MOST EMINENT AUTHORS,

BOTH ANCIENT AND MODERN,

WHO BY THEIR DISCOVERIES OR IMPROVEMENTS HAVE CONTRIBUTED TO THE ADVANCEMENT OF THEM.

BY CHARLES HUTTON, LL.D.

FELLOW OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH, AND OF THE PHILOSOPHICAL SOCIETIES OF HAARLEM AND AMERICA; AND EMERITUS PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

IN TWO VOLUMES:

WITH MANY CUTS AND COPPER-PLATES.

A NEW EDITION,

WITH NUMEROUS ADDITIONS AND IMPROVEMENTS.

VOL. I.

LONDON:

PRINTED FOR THE AUTHOR;

F. C. AND J. RIVINGTON; J. CUTHELL; LAW AND WHITTAKER; LONGMAN, HURST, REES, ORME, AND BROWN; CADELL AND DAVIES; J. MAWMAN; BLACK, PARRY, AND CO; BALDWIN, CRADOCK AND JOY; J. BOOKER; G. AND S. ROBINSON; T. HAMILTON; WALKER AND EDWARDS; JOHN ROBINSON; R. REYNOLDS; AND SIMPKIN AND MARSHALL.

1815.

Charles Hutton, *A mathematical and philosophical dictionary*, London, 1795, vol. 1, p. 91 (p. 96 of revised edition, 1815):

He shewed the universal generation of all the compound or affected equations, by the continual multiplication of so many simple ones; thereby plainly exhibiting to the eye the whole circumstances of the nature, mystery and number of the roots of equations; with the composition and relations of the coefficients of the terms; . . .

Algebra in the 17th century

From 1600 onwards, ‘algebra’ as a set of recipes and techniques began to diverge in two (linked) directions:

- ‘algebra’ as a tool or a language (a.k.a. ‘analysis’ or the ‘analytic art’)
- ‘algebra’ as an object of study in its own right (the ‘theory of equations’)

Descartes on algebra

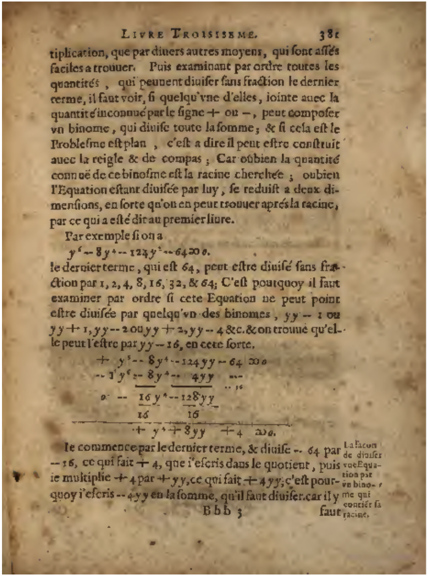
Polynomials feature in Descartes' *La géométrie* (1637), e.g.:

- one example to show that polynomials can be constructed from their roots (influenced by Harriot?);
- 'rule of signs': the number of positive ('true') roots of a polynomial is at most the number of times that the sign changes as we read term-by-term; the number of negative ('false') roots is at most the number of successions of the same sign; for example,

$$x^4 - 4x^3 - 19xx + 106x - 120 = 0$$

  has at most 3 positive roots and at most one negative;
- can always make a transformation to remove the second-highest term.

Descartes on cubics



tiplication, que par diuers autres moyens, qui sont assés faciles a trouuer. Puis examinant par ordre toutes les quantités, qui peuuent diuiser sans fraction le dernier terme, il faut voir, si quelqu'vne d'elles, iointe auec la quantité inconnuë par le signe + ou −, peut composer vn binome, qui diuise toute la somme; & si cela est le Problesme est plan, c'est a dire il peut estre construit auec la reigle & le compas; Car oubien la quantité connuë de ce binosme est la racine cherchée; oubien l'Equation estant diuisée par luy, se reduist a deux dimensions, en sorte qu'on en peut trouuer aprés la racine, par ce qui a esté dit au premier liure.

Par exemple si on a

$y^6 - 8y^4 - 124y^2 - 64$ ∞ 0.

le dernier terme, qui est 64, peut estre diuisé sans fraction par 1, 2, 4, 8, 16, 32, & 64; C'est pourquoy il faut examiner par ordre si cete Equation ne peut point estre diuisée par quelqu'vn des binomes, $yy - 1$ ou $yy + 1$, $yy - 2$ ou $yy + 2$, $yy - 4$ &c. & on trouué qu'elle peut l'estre par $yy - 16$, en cete sorte.

+ $y^6 - 8y^4 - 124yy - 64$ ∞ 0
− $1y^6 - 8y^4 - 4yy$ − − 16
0 − $16y^4 - 128yy$
16 16

+ $y^4 + 8yy + 4$ ∞ 0.

Ie commence par le dernier terme, & diuise − 64 par − 16, ce qui fait + 4, que i'escris dans le quotient, puis ie multiplie + 4 par + yy, ce qui fait + $4yy$; c'est pourquoy i'escris − $4yy$ en la somme, qu'il faut diuiser. car il y faut

La façon de diuiser vne Equation par vn binome qui contient sa racine.

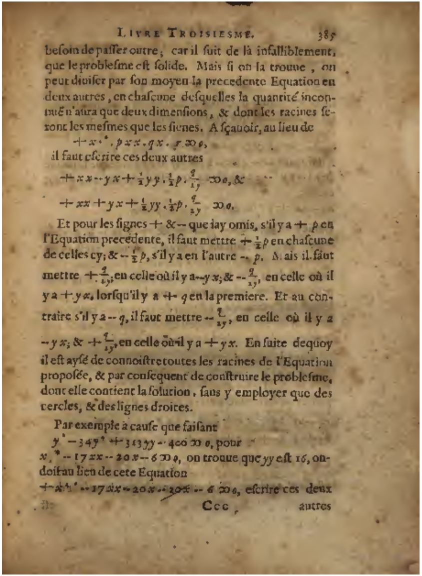
Search for roots of a cubic by examining the factors of the constant term:

if α is such a factor, test whether $x - \alpha$ divides the polynomial.

Examines the example

$$y^6 - 8y^4 - 124y^2 - 64 = 0$$

Descartes on quartics



besoin de passer outre; car il suit de là infalliblement, que le problesme est solide. Mais si on la trouue, on peut diuiser par son moyen la precedente Equation en deux autres, en chascune desquelles la quantité inconnuë n'aura que deux dimensions, & dont les racines seront les mesmes que les siennes. A sçauoir, au lieu de

$+x^4 * .pxx.qx.r \infty 0,$

il faut escrire ces deux autres

$+xx - yx + \frac{1}{2}yy.\frac{1}{2}p.\frac{q}{2y} \infty 0$, &

$+xx + yx + \frac{1}{2}yy.\frac{1}{2}p.\frac{q}{2y} \infty 0.$

Et pour les signes + & − que iay omis, s'il y a $+p$ en l'Equation precedente, il faut mettre $+\frac{1}{2}p$ en chascune de celles cy; & $-\frac{1}{2}p$, s'il y a en l'autre $-p$. Mais il faut mettre $+\frac{q}{2y}$, en celle où il y a $-yx$; & $-\frac{q}{2y}$, en celle où il y a $+yx$, lorsqu'il y a $+q$ en la premiere. Et au contraire s'il y a $-q$, il faut mettre $-\frac{q}{2y}$, en celle où il y a $-yx$; & $+\frac{q}{2y}$, en celle où il y a $+yx$. En suite dequoy il est aysé de connoistre toutes les racines de l'Equation proposée, & par consequent de construire le problesme, dont elle contient la solution, sans y employer que des cercles, & des lignes droites.

Par exemple à cause que faisant

$y^6 - 34y^4 + 313yy - 400 \infty 0$, pour

$x^4 * - 17xx - 20x - 6 \infty 0$, on trouue que yy est 16, on doit au lieu de cete Equation

$+x^4 * - 17xx - 20x - 6 \infty 0$, escrire ces deux

Ccc autres

To solve $+x^4 \star .pxx.qx.r = 0$ (Descartes' notation), that is,

$$x^4 \pm pxx \pm qx \pm r = 0\,,$$

he sought to write the quartic as a product of two quadratics. This led him to

$$y^6 \pm 2py^4 + (pp \pm 4r)yy - qq = 0$$

As in Ferrari's/Cardano's method: a quartic is reduced to a cubic

Summary and a glance ahead

By 1600, general solutions were available for quadratic, cubic and quartic equations — specifically, general solutions in radicals, i.e., solutions constructed from the coefficients of a given polynomial equation via $+$, $-$, $\times$, $\div$, $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, $\ldots$

NB: A solution in radicals may be constructed by ruler and compass.

Spoiler: the general quintic equation is not solvable in radicals.

By the 1770s, mathematicians (notably Lagrange) had come to suspect this, but it was not proved until the 1820s.

So did anything interesting happen in algebra during the 17th and 18th centuries?

A typical 20th-century view

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

Fair point? Or not?

Some 17th-century developments: Hudde's rule (1657)

434 IOHANNIS HUDDENII EPIST. I.

quæro, per Methodum ſuperiùs explicatam, maximum earum communem diviſorem; atque hujus ope æquationem Propoſitam toties divido, quoties id fieri poteſt.

Exempli gratiâ, proponatur hæc æquatio $x^3 - 4xx + 5x - 2 \infty 0$, in qua duæ ſunt æquales radices. Multiplico ergo ipſam per Arithmeticam Progreſſionem qualemcunque, hoc eſt, cujus incrementum vel decrementum ſit vel 1, vel 2, vel 3, vel alius quilibet numerus; & cujus primus terminus ſit vel 0, vel +, vel − quam 0: Ita ut ſemper ejus ope talis terminus æquationis tolli poſſit, qualem quis voluerit, collocando tantùm ſub eo 0.

Ut ſi, exempli causâ, ultimum ejus terminum auferre velim, multiplicatio fieri poteſt ipſius $x^3 - 4xx + 5x - 2 \infty 0$
per hanc progreſſionem $3.\quad 2.\quad 1.\quad 0$
fietque $3x^3 - 8xx + 5x \; * \; \infty 0$.

Maxima autem communis diviſor hujus & Propoſitæ æquationis eſt $x - 1 \infty 0$, per quam Propoſita bis dividi poteſt; ita ut ejuſdem radices ſint 1, 1, & 2.

Sic ſi cupiam 1mum æquationis terminum auferre, multiplicatio inſtitui poteſt ipſius $x^3 - 4xx + 5x - 2 \infty 0$
per hanc progreſſionem $0.\quad 1.\quad 2.\quad 3.$
& fit $* \; -4xx + 10x - 6 \infty 0$.

Cujus quidem ac Propoſitæ æquationis maximus communis diviſor, ut antea, eſt $x - 1 \infty 0$.

Similiter ſi 2dum terminum tollere lubeat, multiplicatio fieri poteſt, hoc pacto: $x^3 - 4xx + 5x - 2 \infty 0$
$+1.\quad 0.\quad -1.\quad -2$
& prodibit $x^3 \; * \; -5x + 4 \infty 0$.

Cujus item & Propoſitæ maximus communis diviſor eſt $x - 1 \infty 0$.

Ubi notandum, non neceſſarium eſſe, ſemper uti Progreſſione cujus exceſſus ſit 1, quanquam ea communiter ſit optima.

Published 1659 as an addendum to van Schooten's Latin translation of Descartes' *La géométrie*:

$x^3 - 4xx + 5x - 2 = 0$ has a double root $x = 1$;

multiply the terms of the equation by numbers in arithmetic progression:

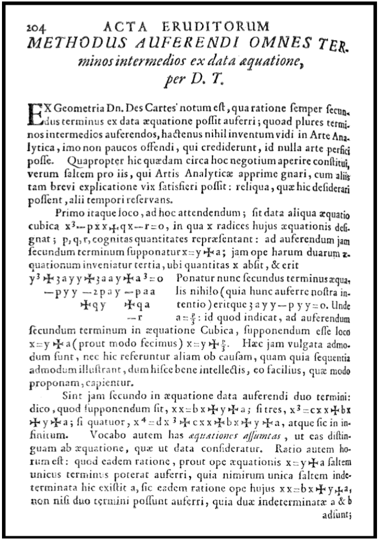
$3x^3 - 8xx + 5x = 0$ also has a double root $x = 1$,

as does $-4xx + 10x - 6 = 0$.

(Modern form of rule: if r is a double root of $f(x) = 0$, then it is a root of $f'(x) = 0$ also.)

See *Mathematics emerging*, §12.2.2.

Some 17th-century developments: Tschirnhaus transformations (1683)



METHODUS AUFERENDI OMNES TERminos intermedios ex data æquatione, per D. T.

EX Geometria Dn. Des Cartes' notum est, qua ratione semper secundus terminus ex data æquatione possit auferri; quoad plures terminos intermedios auferendos, hactenus nihil inventum vidi in Arte Analytica, imo non paucos offendi, qui crediderunt, id nulla arte perfici posse. Quapropter hic quædam circa hoc negotium aperire constitui, verum saltem pro iis, qui Artis Analyticæ apprime gnari, cum aliis tam brevi explicatione vix satisfieri possit: reliqua, quæ hic desiderari possent, alii tempori reservans.

Primo itaque loco, ad hoc attendendum; sit data aliqua æquatio cubica $x^3 - pxx + qx - r = 0$, in qua x radices hujus æquationis designat; p, q, r, cognitas quantitates repræsentant: ad auferendum jam secundum terminum supponatur $x = y + a$; jam ope harum duarum æquationum inveniatur tertia, ubi quantitas x absit, & erit

$$\begin{array}{llll} y^3 + 3ayy & + 3aay & + a^3 & = 0 \\ \quad - pyy & - 2pay & - paa & \\ & + qy & + qa & \\ & & - r & \end{array}$$

Ponatur nunc secundus terminus æqualis nihilo (quia hunc auferre nostra intentio) eritque $3ayy - pyy = 0$. Unde $a = \frac{p}{3}$: id quod indicat, ad auferendum secundum terminum in æquatione Cubica, supponendum esse loco $x = y + a$ (prout modo fecimus) $x = y + \frac{p}{3}$. Hæc jam vulgata admodum sunt, nec hic referuntur aliam ob causam, quam quia sequentia admodum illustrant, dum hisce bene intellectis, eo facilius, quæ modo proponam, capientur.

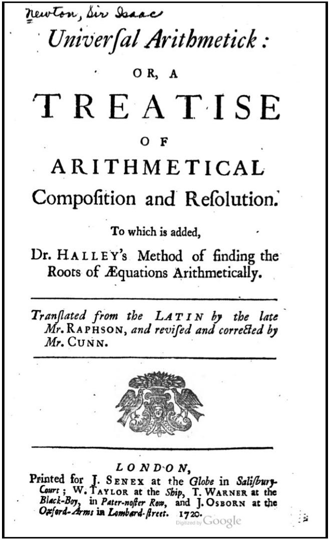
Sint jam secundo in æquatione data auferendi duo termini: dico, quod supponendum sit, $xx = bx + y + a$; si tres, $x^3 = cxx + bx + y + a$; si quatuor, $x^4 = dx^3 + cxx + bx + y + a$, atque sic in infinitum. Vocabo autem has *æquationes assumtas*, ut eas distinguam ab æquatione, quæ ut data consideratur. Ratio autem horum est: quod eadem ratione, prout ope æquationis $x = y + a$ saltem unicus terminus poterat auferri, quia nimirum unica saltem indeterminata hic existit a, sic eadem ratione ope hujus $xx = bx + y + a$, non nisi duo termini possunt auferri, quia duæ indeterminatæ a & b adsunt;

For an equation $x^3 - px^2 + qx - r = 0$

- to remove one term put $x = y + a$ (where $a = p/3$)
- can we remove both the middle terms?
- to remove two terms put $x^2 = bx + y + a$

See *Mathematics emerging*, §12.2.3.

An 18th-century development: Newton's *Arithmetica universalis* (1707)



newton, Sir Isaac

Univerſal Arithmetick :

OR, A

TREATISE

OF

ARITHMETICAL

Compoſition and Reſolution.

To which is added,

Dr. HALLEY's Method of finding the Roots of Æquations Arithmetically.

Tranſlated from the LATIN *by the late* Mr. RAPHSON, *and reviſed and correċted by* Mr. CUNN.

LONDON,

Printed for J. SENEX at the *Globe* in *Saliſbury-Court*; W. TAYLOR at the *Ship*, T. WARNER at the *Black-Boy*, in *Pater-noſter Row*, and J. OSBORN at the *Oxford-Arms* in *Lombard-ſtreet*. 1720.

Rules for sums of powers of roots of

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \cdots = 0$$

$$\begin{aligned} \text{sum of roots} &= p \\ \text{sum of roots}^2 &= pa - 2q \\ \text{sum of roots}^3 &= pb - qa + 3r \\ \text{sum of roots}^4 &= pc - qb + ra - 4s \end{aligned}$$

Developments of the 17th and 18th centuries

- Symbolic notation
- Understanding of the structure of polynomials
- ... of the number and nature of their roots
- ... of the relationship between roots and coefficients
- ... of how to manipulate them
- ... of how to solve them numerically
- The leaving behind of geometric intuition?

Some 18th-century theory of equations

Recall:

- cubic equations can be solved by means of quadratics
- quartic equations can be solved by means of cubics

Some 18th-century theory of equations

Recall:

- quadratic equations can be solved by means of linear equations
- cubic equations can be solved by means of quadratics
- quartic equations can be solved by means of cubics

The ‘reduced’ or ‘resolvent’ equation:

- for cubics, the reduced equation is of degree 2
- for quartics, the reduced equation is of degree 3
- for quintics, the reduced equation is of degree ?

Some 18th-century hypotheses

Euler's hypothesis (1733):

- for an equation of degree n the degree of the reduced equation will be $n - 1$

Bézout's hypothesis (1764):

- for an equation of degree n the degree of the reduced equation will in general be $n!$
- though always reducible to $(n - 1)!$
- possibly further reducible to $(n - 2)!$

Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/1):

Examined all known methods of solving

- quadratics: the well-known solution
- cubics: methods of Cardano, Tschirnhaus, Euler, Bézout
- quartics: methods of Cardano, Descartes, Tschirnhaus, Euler, Bézout

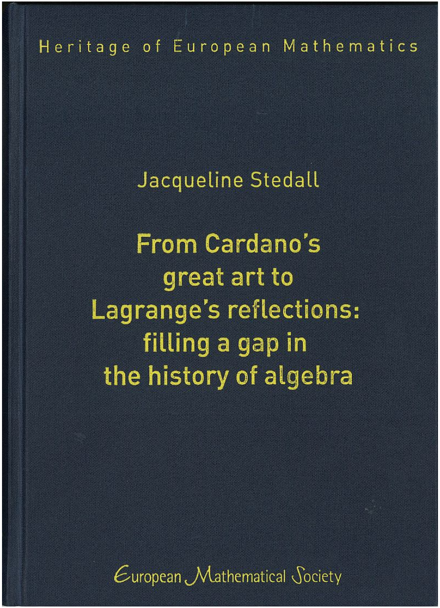
seeking to identify a uniform method that could be extended to higher degree

A typical 20th-century view revisited

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

Filling a gap in the history of algebra (2011)



The hitherto untold story of the slow and halting journey from Cardano's solution recipes to Lagrange's sophisticated considerations of permutations and functions of the roots of equations ... [Preface]

From Stedall's preface:

This assertion . . . from Nový quoted above, betrays yet another fundamental shortcoming of several earlier accounts, a view that mathematics somehow progresses only by means of 'great and significant works' and 'substantial changes'. Fortunately, the truth is far more subtle and far more interesting: mathematics is the result of a cumulative endeavour to which many people have contributed, and not only through their successes but through half-formed thoughts, tentative proposals, partially worked solutions, and even outright failure. No part of mathematics came to birth in the form that it now appears in a modern textbook: mathematical creativity can be slow, sometimes messy, often frustrating.