River Flow

Water in the earth is found on land, in the oceans and in the atmosphere. The main science of water resources engineering is *hydrology*, and deals with the occurrence, distribution, movement and properties of water on earth. Hydrology is primarily concerned with water on land and in the atmosphere, from its deposition as rainfall or snowfall, to its flow to the oceans and its evaporation back into the atmosphere.

The hydrologic cycle

The hydrologic cycle is defined as the pathway of water as it moves in its various phases through the atmosphere, to the land surface, over and through the land, to the ocean and back to the atmosphere. This movement is shown diagrammatically in Figure 7.1.1 below, where the relative magnitudes of various hydrologic processes are given in units relative to the value of 100 for the rate of precipitation on land. These rates are based on global annual averages. Starting with evaporation of water from the oceans and driven by the heat of the sun, evaporated water in the form of water vapour rises by convection; condenses in the atmosphere to form clouds, and precipitates on the land and ocean surfaces as predominantly rain or snow. Precipitation on land surfaces is partially intercepted by surface vegetation, partially stored in surface depressions, partially infiltrated into the ground and partially flows over land into drainage channels and rivers that finally lead back to the ocean.

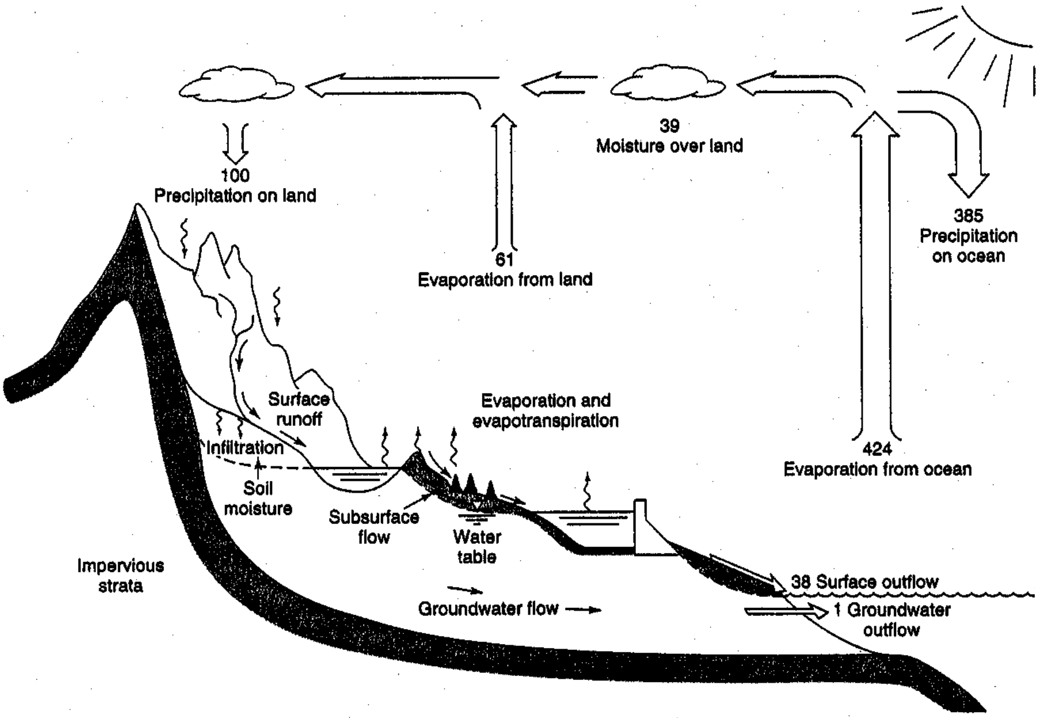


Figure 7.1.1 Hydrologic cycle with global annual average water balance given in units relative to a value of 100 for the rate of precipitation on land (from Chow et al. (1988)).

(Source: Chin 2000)

Aquifers refer to water stored in the soil under saturated conditions, i.e. there is no soil air. *Confined aquifers* refers to water stored in between two impermeable layers, while *unconfined aquifers* sit on a lower impermeable layer with their upper boundary defined by the water table. The *water table* is the 2D surface upon which the water pressure is atmospheric. Below this surface the ground is said to be saturated and the water there is usually referred to as groundwater. Above the water table the ground is unsaturated and water there is usually referred to as soil moisture or soil water. The unsaturated zone is also known as the vadose zone.

On a global scale the distribution of water resources on the earth is shown in the following Table. It is clear that the vast majority of the earth's water is in the oceans and that most of the fresh water is stored in the ground or in the polar ice caps. The amount of water stored in the atmosphere is relatively small, however it is the flux of water in and out of the atmosphere that dominates the hydrologic cycle.

Item	Volume ($\times 10^3$ km^3)	Percent total water (%)	Percent fresh water (%)
Oceans	1,338,000.	96.5	
Ground water			
Fresh	10,530.	0.76	30.1
Saline	12,870.	0.93	
Soil moisture	16.5	0.0012	0.05
Polar ice	24,023.5	1.7	68.6
Other ice and snow	340.6	0.025	1.0
Lakes			
Fresh	91.	0.007	0.26
Saline	85.4	0.006	
Marshes	11.47	0.0008	0.03
Rivers	2.12	0.0002	0.006
Biological water	1.12	0.0001	0.003
Atmospheric water	12.9	0.001	0.04
Total water	1,385,984.61	100.	
Fresh water	35,029.21	2.5	100.

Source: USSR National Committee for the International Hydrological Decade (1978).

The estimated annual average fluxes of precipitation, evaporation and runoff within the global hydrologic cycle are as follows

Component	Oceanic flux (mm/year)	Terrestrial flux (mm/year)
Precipitation	1,270	800
Evaporation	1,400	484
Runoff to ocean (rivers plus ground water)	—	316

Source: USSR National Committee for the International Hydrological Decade (1978).

(Source: Chin 2000)

The above Table says that the global average precipitation on land is of the order of 800 mm/yr, of which 484 mm/yr is returned to the atmosphere from evaporation and 316 mm/yr is returned to the ocean from surface runoff.

Typical residence times for water in the atmosphere is about one week, while moisture in the soil can have residence times which vary from weeks to centuries! Rivers tend to have a mean residence time of the order of 13 days.

Why are we interested in the flow of water over the land surface?

- evolution and erosion of hillslopes, landforms
- development of drainage networks
- river evolution and meandering
- river responses to heavy rainfall, i.e. flood prediction modelling, discharge hydrographs
- movement of waves in rivers, canals etc

It is usual to distinguish between two major types of free surface flow:

1. sheet flow or overland flow, which generally occurs under heavy rainfall and then feeds into streams

2. flow that occurs in larger permanent open channels, i.e. rivers, streams canals etc.

Both types of flow are usually unsteady and spatially varied. We will focus mainly on river flow but also show how the same formulation easily applies over to shallow overland flow.

Hydrographs

Hydrographs represent the relationship between flow rate, discharge and time at a particular location on a stream. It is an integration of the topographic and climatic characteristics that effect the relation between rainfall on, and runoff from a drainage basin.

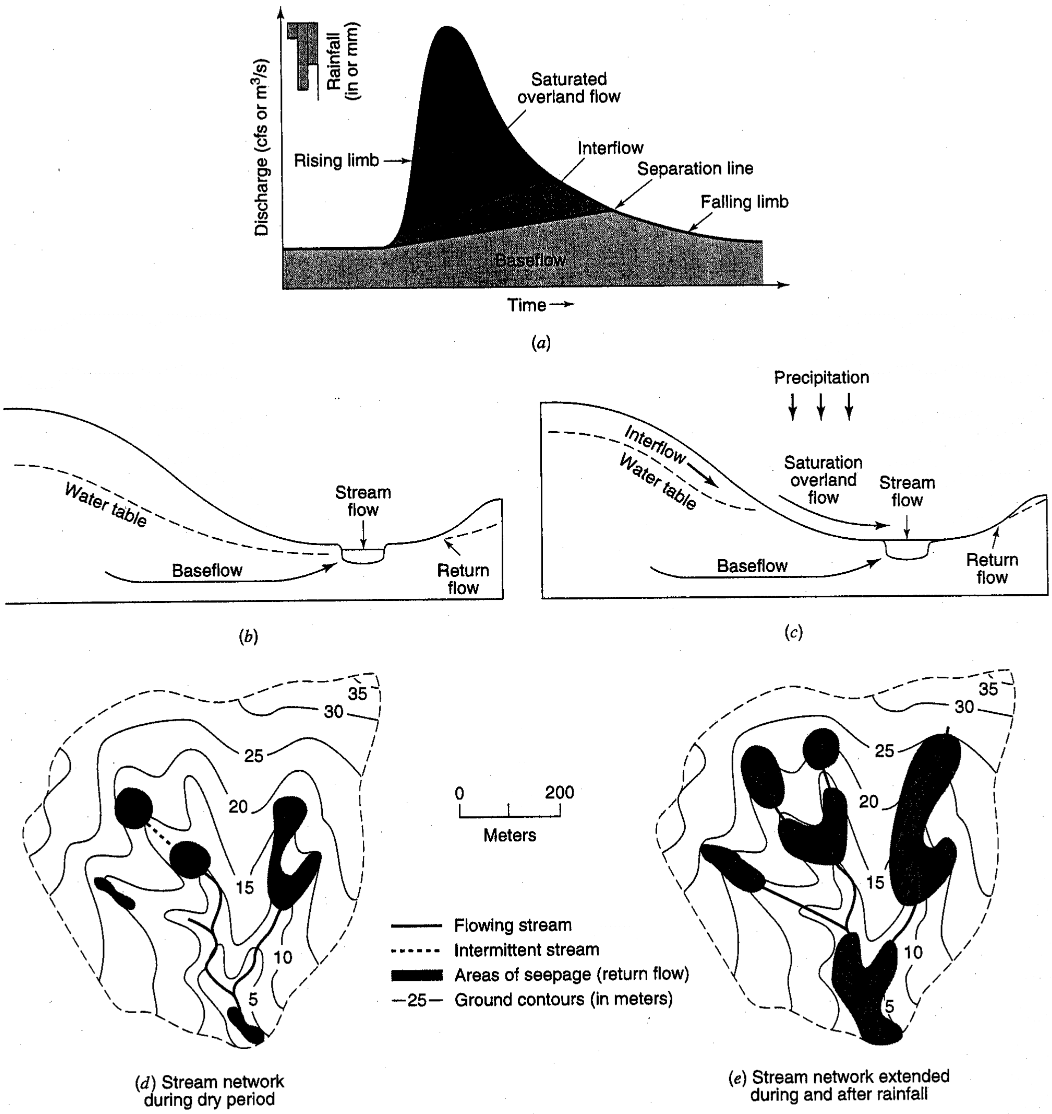


Figure 2.1. (a) Separation of sources of streamflow on an idealized hydrograph; (b) Sources of streamflow on a hillslope profile during a dry period; (c) During a rainfall event; (d) Stream network during a dry period; (d) Stream network extended during and after rainfall (from Mosley and McKerchen, 1993)

Figure 2.1 (a) shows a discharge hydrograph, or volumetric flow rate against time from a drainage basin, or a river. Note that the discharge curve is subdivided into various components, overland flow, and subsurface flow which is separated into interflow and baseflow.

Overland Flow:- flow resulting directly from surface runoff

Base flow:- quasi-independent of rainfall and is due to the flow of groundwater into the drainage channel. It depends on the difference in elevation between the groundwater surface and the water surface in the drainage channel.

Interflow:- this is the inflow to the drainage channel due to water flow between the land surface and the water table.

The direct runoff from a storm event (overland flow) is added to the base flow and interflow to give the discharge hydrograph for the drainage channel. Figures 2.1 (b) and (c) show how a water table rises and falls in response to rainfall, while (c) and (d) shows how a channel network expands and contracts in response to rainfall.

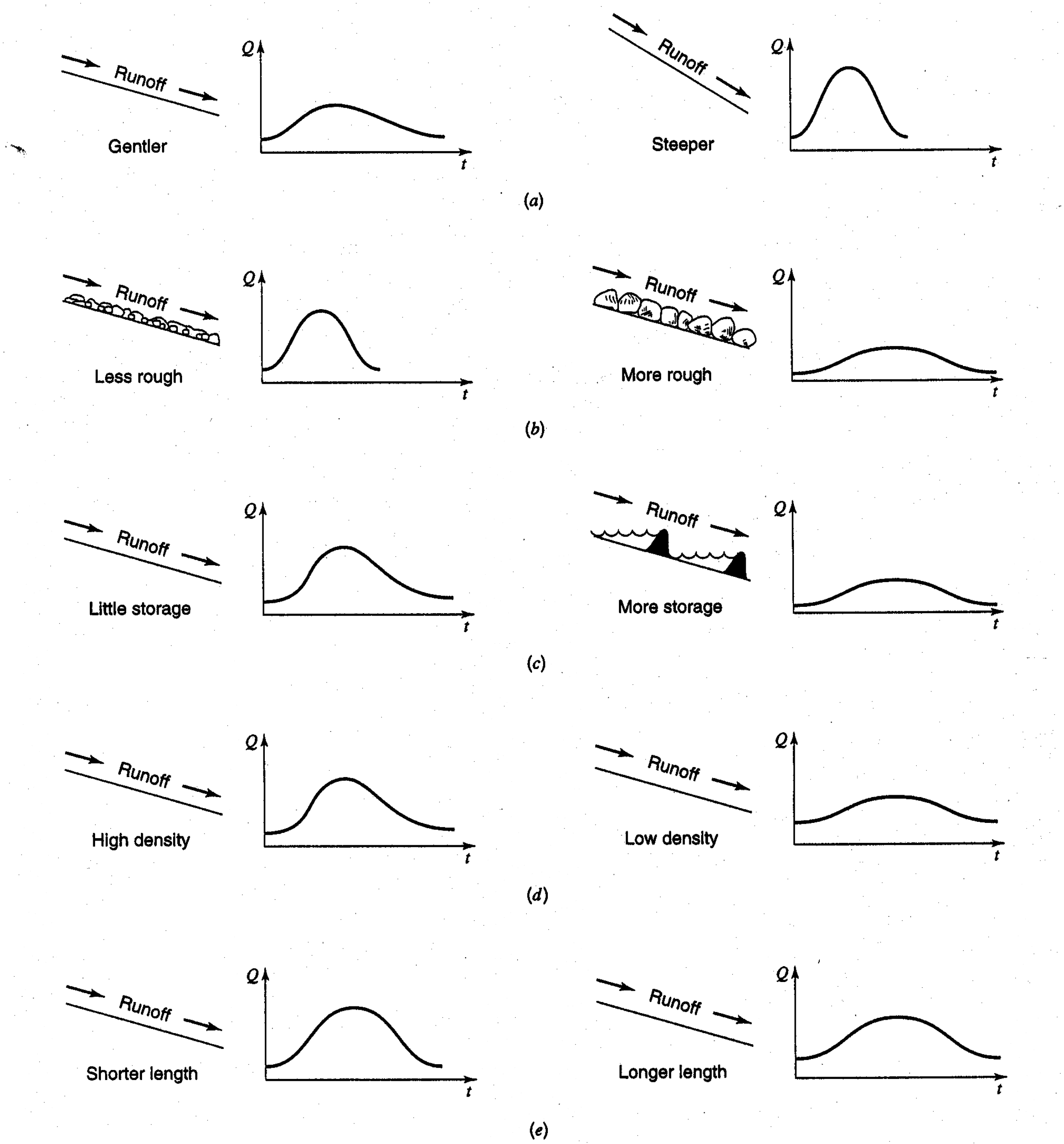


Figure 2.2. Effects of basin characteristics on the flood hydrograph. (a) Relationship of slope to peak discharge; (b) Relationship of hydraulic roughness to runoff; (c) Relationship of

storage to runoff; (d) Relationship of drainage density to runoff; (e) Relationship of channel length to runoff (from Masch 1984)

Figure 2.2 demonstrates how the various topographic and hydraulic characteristic of a drainage basin can effect the discharge hydrograph. These factors include

- drainage area
- slope
- hydraulic roughness
- natural and channel storage
- stream length
- channel density
- antecedent moisture conditions
- vegetation
- channel modification

Finally Figure 2.3 illustrates the effects of storm shape, size and movement across a catchment in relation to the resulting catchment discharge hydrograph.

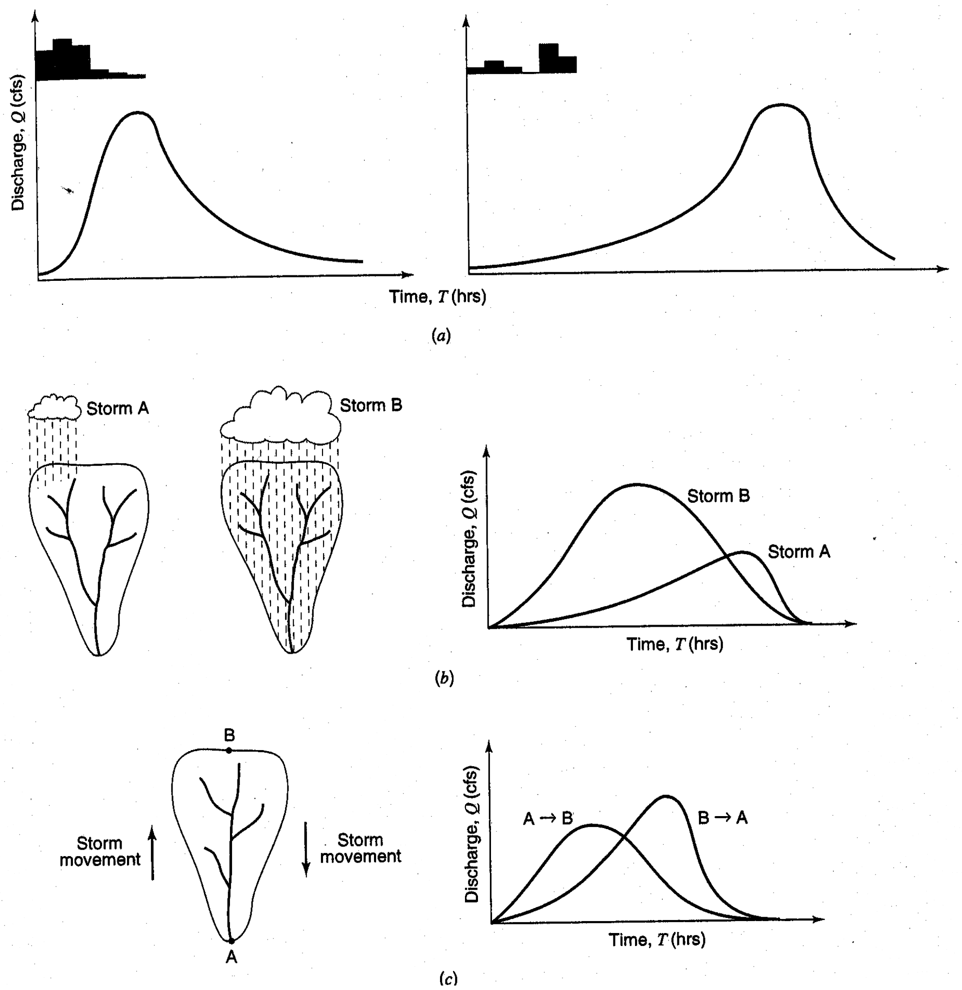


Figure 2.3. Effects of storm shape, size and movement on surface runoff. (a) Effect of time variation of rainfall intensity on runoff; (b) Effect of storm size on runoff; (c) Effect of storm movement on surface runoff (from Masch 1984)

Types of rivers

Braided rivers consist of a network of small channels separated by small and often temporary islands called braid bars. Braided channels are also typical of river deltas and alluvial fans.

Meandering rivers results from erosion, transportation and deposition of sediment along its banks.

Mass Conservation

Consider a straight section of river and define a control volume (CV) of length Δs and define the following:

Q = volume flux of water down the river, (m^3 s^{-1})
A = cross section area of the river, (m^2)
x = downstream distance, (m)
ρ = fluid density, (kg m^{-3})

Mass conservation through a control volume states that

Rate of increase of mass in an element = net rate of flow of mass into the element

$\dfrac{\partial(\rho A)}{\partial t}\Delta x$ = the rate of change of mass per unit width stored in CV of size Δx

$\rho Q - \dfrac{\partial(\rho Q)}{\partial x}\dfrac{\Delta x}{2}$ = volumetric flow rate across upstream face of the CV

$\rho Q + \dfrac{\partial(\rho Q)}{\partial x}\dfrac{\Delta x}{2}$ = volumetric flow rate across downstream face of the CV

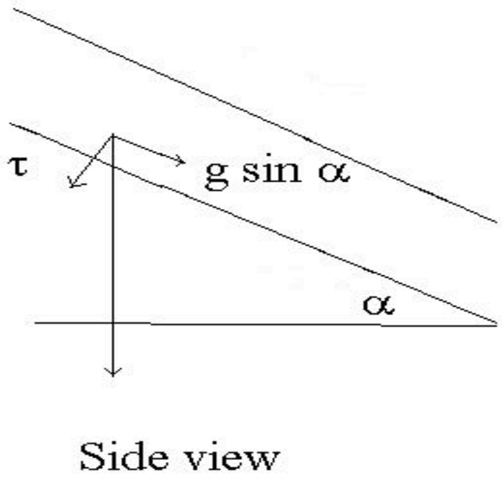
therefore mass conservation for an incompressible fluid becomes

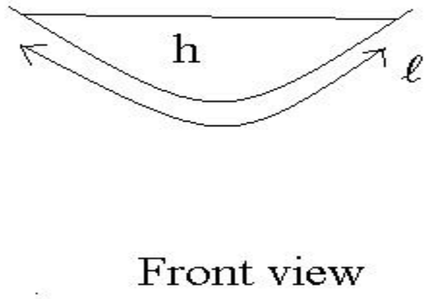
$$\frac{\partial(\rho A)}{\partial t}\Delta x = \left[\rho Q - \frac{\partial(\rho Q)}{\partial x}\frac{\Delta x}{2}\right] - \left[\rho Q + \frac{\partial(\rho Q)}{\partial x}\frac{\Delta x}{2}\right], \tag{1}$$

or (since taking as incompressible)

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0. \tag{2}$$

To use (2) we need a relationship between Q and A. Consider a force balance on slowly varying flow, i.e. we can neglect acceleration, then the weight of fluid is balanced against the boundary shear stress





where α = slope angle, ℓ = wetted perimeter length and τ = shear stress. From the force balance

$$\tau\ell = \rho g A S$$

or

$$\tau = \rho g R S \tag{3}$$

where $R = A/\ell$ is the hydraulic radius and $S = \sin\alpha$.

Turbulent Flow

The Reynolds number is defined as

$$\mathrm{Re} = \frac{uh}{\nu},$$

where u = mean velocity, h = mean depth and ν = kinematic viscosity (10^{-6} m^2 s^{-1} = μ/ρ , μ = dynamic viscosity). For Re > 1000, free surface flow in nature may generally be considered fully turbulent.

To model turbulent flow, a empirically derived friction law is commonly used, $\tau = fgu^2$ where f is a friction factor which typical values ≤ 0.03. [Note slightly different definitions arise here as it is common to see $\tau = \frac{f}{2} gu^2$ in the literature]. Thus equating with (3), we obtain

$$\begin{aligned} fgu^2 &= \rho gRS \\ \Rightarrow\ u &= CR^{1/2}S^{1/2}\ . \end{aligned} \tag{4}$$

Equation (4) is known as Chezy's law (1775) with $C = (g/f)^{1/2}$ known as the Chezy coefficient. For most present applications, Manning's equation is often used instead of Chezy's law for open channel computations. This equation was developed in 1890 and uses a roughness coefficient n (as opposed to a friction factor f), and is given by

$$u = \frac{1}{n} R^{2/3} S^{1/2}. \tag{5}$$

NOTE that n has units of m$^{-1/3}$ s and ranges from approximately 0.01 for smooth to 0.1 for rough channels. From (4) and (5) one can deduce that most empirical equations derived from steady uniform flow measurements are of the form

$$u = C_r R^a S^b, \tag{6}$$

where C_r is a resistance factor and a, b are constants.

Now given that $Q = uA$, $R = A/\ell$, then from (6)

$$Q = C_r \frac{S^b}{\ell^a} A^{a+1} = f(A), \tag{7}$$

where the effect of channel shape, ie semicircular, triangular (notch), wide or trapezoidal comes directly through the hydraulic radius R.

Effect of Channel Shape

(a) Semicircular (if full)

$$A = \frac{\pi r^2}{2}, \quad \ell = \pi r, \ \Rightarrow \ A = \beta^2 \ell^2, \ \ \beta^2 = \frac{1}{2\pi}, \text{ therefore } R = \frac{A}{\ell} = \beta\sqrt{A} \ .$$

(b) Triangular or notch (θ = angle between water surface and side of notch)

$$A = \beta^2 \ell^2, \ \ \beta^2 = \frac{1}{8}\sin(2\theta), \quad \text{, therefore } R = \frac{A}{\ell} = \beta\sqrt{A} \ .$$

(c) Wide rectangular canal of width w containing water of depth h

$$A = wh, \ \ \ell = w + 2h \ \ \text{, therefore } R = \frac{A}{\ell} = \frac{A}{w} \ \text{ since } w >> 2h \ .$$

As $Q = CR^{1/2} S^{1/2} A$ for Chezy and $Q = R^{2/3} S^{1/2} A / n$ for Mannings, then we have the following $Q(A)$ relationships for various shaped channels:

	Chezy	Mannings
Circular/notch $R = \beta\sqrt{A}$	: $A^{5/4}$	: $A^{4/3}$
Wide canal $R = A/w$	: $A^{3/2}$	: $A^{5/3}$

It is clear from the above table that we can write the general relationship for $Q(A)$ as

$$Q(A) = \frac{c}{m+1} A^{m+1} \quad m > 0, \tag{8}$$

and in particular for a Chezy canal we have

$$Q(A) = \left(\frac{g}{f}\right)^{1/2} \left(\frac{A}{w}\right)^{1/2} S^{1/2} A = \left(\frac{gS}{wf}\right)^{1/2} A^{3/2} \ , \tag{9}$$

where m = 1/2 and $c = (3/2)(gS/wf)^{1/2}$.

Slowly Varying Flow
When the effects of acceleration and hydrostatic pressure gradients can be neglected, the flow is described by

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0, \tag{10}$$

with $Q(A)=\dfrac{c}{m+1}A^{m+1}$, thus

$$\frac{\partial A}{\partial t}+cA^{m}\frac{\partial A}{\partial x}=0. \tag{11}$$

Consider a general solution to (11) for an arbitrary initial condition

$$t=0 \quad A=A_{o}(x). \tag{12}$$

By the method of characteristics we can write

$$\frac{dt}{1}=\frac{dx}{cA^{m}}=\frac{dA}{0}, \tag{13}$$

Using σ to parameterize the characteristics, then from (13) A must be constant along a characteristic and given by

$$A=A_{o}(\sigma). \tag{14}$$

The characteristics are given from the first and second terms of (13) as

$$x-\sigma=cA_{o}(\sigma)^{m}, \tag{15}$$

then eliminating σ between (14) and (15) gives the solution as

$$A=A_{o}(x-cA^{m}t). \tag{16}$$

Shocks and Flood Hydrograph (see Andrew's book, Mathematical Geoscience, section 4.3 pp 229 – 229))

Hillslope Flow

Shallow overland flow is quite different to river flow, for example flow depths are of the order of 0.5 to 5 cm and flow velocities are only usually around 2 cm/s. However these flows are still described essentially through (11). Overland flows across farmlands can often result in the transport of chemical (pesticides, fertilizers) or virus laden sediment into surrounding surface water bodies (lakes, rivers, canals etc). To estimating the pollutant transport rates into these water bodies you first need to be able to calculate the water flow field. For overland flow

though, source and sink terms need to be added to (11) which account for rainfall rate, P¸ (being the source of water) and infiltration rate, I, through the soil surface (loss of water).
Consider flow down a hillslope and denote x as the downslope distance with $x = 0$ being the hill apex. Define $t = 0$ as the time surface runoff first appears from a rainfall rate $P(t)$, assuming a constant width w for the hillslope and a Manning's friction law, then $R = h$, $u = S^{1/2}h^{2/3}/n$, and $Q = uA = wS^{1/2}h^{5/3}/n$. Thus we wish to solve the following problem for $h(x,t)$;

$$\frac{\partial h}{\partial t} + ch^m \frac{\partial h}{\partial x} = P - I = E(t), \tag{17}$$

subject to the initial and boundary conditions

$$\begin{aligned} &t = 0,\ \ x > 0,\ \ h = 0 \\ &t > 0,\ \ x = 0,\ \ h = 0\ \ . \end{aligned} \tag{18}$$

In (17), $m = 2/3$, $c = (m+1)S^{1/2}/n$,and E is the excess rainfall rate, being the difference between rainfall and infiltration rates. In the solution to follow we will assume that $E(t)$ >0 for all time, though this is not true if rainfall stops while there is still surface water, hence E < 0 in this case.

By the method of characteristics we can write

$$\frac{dt}{1} = \frac{dx}{ch^m} = \frac{dh}{E(t)}. \tag{19}$$

The initial data is satisfied through the characteristics which emanate from the x axis which are parameterized through x_o. The boundary condition is satisfied by characteristics emanating from the t axis which are parameterized through t_o. Thus the solution domain is divided into two regions by the characteristic coming out of the origin and given by $x_o = t_o = 0$.

E(t) > 0 for all t

Region (i) x axis characteristics

In this region the (19) is solved subject to

$$t = 0,\ \ x = x_o,\ \ h = 0. \tag{20}$$

Taking the first and last terms of (19) gives

$$h(t) = \int_0^t E(t')\,dt'. \tag{21}$$

The characteristics are then given by taken the first and second terms of (19) along with (21) as

$$x - x_0 = c\int_0^t h^m dt = c\int_0^t \left[\int_0^{\bar{t}} E(t')\,dt'\right]^m d\bar{t}\ . \tag{22}$$

Region (ii) t axis characteristics
In this region the (19) is solved subject to

$$t = t_0, \;\; x = 0, \;\; h = 0. \tag{23}$$

Taking the second and last terms of (19) gives

$$h(t_o, t) = \int_{t_o}^{t} E(t')\, dt'. \tag{24}$$

The characteristics are then given by taken the first and second terms of (19) along with (24) as

$$x(t_o, t) = c\int_{t_o}^{t} h^m dt = c\int_{t_o}^{t} \left[\int_{t_o}^{\bar{t}} E(t')\, dt' \right]^m d\bar{t}\ . \tag{25}$$

with the parameter t_o in the range $0 \le t_o \le t$. The boundary condition in (18) is satisfied through $t_o = t$ while for $t_o = 0$, (24) and (25) match onto (21) and (22) with x_o=0 respectively. In region (ii) h is both x and t dependent and is given parametrically through t_o while in region (i) h is only t dependent.

E (t) ≥ 0 for 0 ≤ t ≤ t* and E(t) < 0 for t > t*

[HINT] This occurs when rainfall has stopped but infiltration can still continue as long as there is surface runoff. Once $t > t^*$, $h = 0$ no longer occurs at $x = 0$ but moves downhill. The position of the edge of the drying surface is found from (24) and (25) by considering what happens in (24) when $t > t^*$ or $E < 0$. For $t < t^*$, the solution is still given as above.

Momentum Conservation

Momentum conservation through a control volume states that

Rate of increase of momentum of a CV = net rate of flow of momentum out of CV + sum of forces on fluid particle

Now there are three forces acting on the CV:

$$\sum F = F_p + F_g + F_f \tag{26}$$

where F_p is the unbalanced hydrostatic pressure force, F_g is the gravity force and F_f is the friction force.

With $Q = uA$, then

$\dfrac{\partial(\rho u A)}{\partial t}\Delta x$ = the rate of change of momentum per unit width stored in CV of size Δx

$\rho u Q - \dfrac{\partial(\rho u Q)}{\partial x}\dfrac{\Delta x}{2}$ = momentum flow rate across upstream face of the CV

$\rho u Q + \dfrac{\partial(\rho Q)}{\partial x}\dfrac{\Delta x}{2}$ = mass flow rate across downstream face of the CV

Therefore the net flow of momentum is

$$\left[\rho u Q - \frac{\partial(\rho u Q)}{\partial x}\frac{\Delta x}{2}\right] - \left[\rho u Q + \frac{\partial(\rho u Q)}{\partial x}\frac{\Delta x}{2}\right] = -\frac{\partial(\rho u^2 A)}{\partial x}\Delta x \tag{27}$$

(i) *Hydrostatic pressure Force*

Define an average pressure $\overline{p}$ for the cross section of the stream as

$$\overline{p}A = \int_y \int_0^h \rho g z \, dz \, dy = \frac{1}{2}\rho g \int_y h^2 \, dy$$

where y is the distance across the stream, then

$\overline{p}A - \dfrac{\partial(\overline{p}A)}{\partial x}\dfrac{\Delta x}{2}$ = hydrostatic pressure force on upstream face of the CV

$-\left[\overline{p}A + \dfrac{\partial(\overline{p}A)}{\partial x}\dfrac{\Delta x}{2}\right]$ = hydrostatic pressure force on downstream face of the CV.

Therefore

$$\begin{aligned} F_p &= \left[\overline{p}A - \frac{\partial(\overline{p}A)}{\partial x}\frac{\Delta x}{2}\right] - \left[\overline{p}A + \frac{\partial(\overline{p}A)}{\partial x}\frac{\Delta x}{2}\right] \\ &= -\frac{\partial(\overline{p}A)}{\partial x}\Delta x \\ &= -\rho g \int_y h \frac{\partial h}{\partial x} dy \ . \end{aligned} \tag{28}$$

If we assume that $\partial h / \partial x$ does not vary with y, then we can write (28) as

$$F_p = -\rho g \int_y h \frac{\partial h}{\partial x} dy = -\rho g A \frac{\partial \overline{h}}{\partial x}. \tag{29}$$

(ii) *Gravity Force*

$$\begin{aligned} F_g &= mg \sin\theta \\ &= (\rho A \Delta x) g S \ , \end{aligned} \tag{30}$$

with $S = \sin\theta$.

(iii) *Friction Force*

These are created by the shear stress along the bottom of the CV and are given by

$$F_f = -\ell\tau\,\Delta x\,. \tag{31}$$

Thus momentum conservation becomes

$$\frac{\partial(\rho u A)}{\partial t}\Delta x = -\frac{\partial(\rho u^2 A)}{\partial x}\Delta x - \rho g A\frac{\partial \bar{h}}{\partial x}\Delta x + \rho g A S\Delta x - \ell\tau\,\Delta x\,, \tag{32}$$

or

$$\frac{\partial(uA)}{\partial t} + \frac{\partial(u^2 A)}{\partial x} = gAS - \frac{\ell\tau}{\rho} - gA\frac{\partial \bar{h}}{\partial x}\ . \tag{33}$$

By expanding the derivative terms on the left hand side of (33) and using mass conservation [(2)], then (33) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = gS - \frac{\tau}{\rho R} - g\frac{\partial \bar{h}}{\partial x}\ , \tag{34}$$

which with (2), i.e.

$$\frac{\partial A}{\partial t} + \frac{\partial(uA)}{\partial x} = 0, \tag{35}$$

Equations (34) and (35) are known as the St Venant equations.

Consider the friction term and the pressure gradient:

From a Chezy friction law $\tau = f\rho u^2$ and for a Mannings' law $\tau = \rho g n^2 \dfrac{u^2}{R^{1/3}}$, however it is common in the hydraulics literature to see τ written as $\tau = \rho g R S_f$ where S_f is known as the friction slope. Thus $\ell\tau / \rho A = \tau / \rho R = g S_f$ and the gravity + friction = $g(S - S_f)$ with

$$S_f = \frac{u^2}{RC^2},\quad C = \left(\frac{g}{f}\right)^{1/2} \quad \text{Chezy}$$

$$S_f = \frac{n^2 u^2}{R^{4/3}},\quad \text{Mannings'}\ .$$

Under slowly varying flow only gravity and friction dominate the momentum equation, thus to first order the solution of the momentum equation is given by $S = S_f$. This then gives the friction laws for Chezy and Mannings as before by replacing S_f by S.

Let us now consider the forms that the friction and pressure gradient terms take:

Canal: $R = \dfrac{A}{w}$, $\bar{h} = \dfrac{A}{w}$

Notch: $R = \beta\sqrt{A}$, $\bar{h} = \dfrac{h}{2} \propto \sqrt{A}$, $h = \dfrac{\ell}{2}\sin(\theta)$, $\ell = \sqrt{A}\,/\,\beta$, $A = \dfrac{\ell^2}{8}\sin(2\theta)$
(triangular)

Chezy : $\tau = f\rho u^2$ Mannings : $\tau = \rho g n^2 \dfrac{u^2}{R^{1/3}}$	Friction $\dfrac{\ell\tau}{\rho A} = \dfrac{\tau}{\rho R}$	Pressure gradient $g\dfrac{\partial \bar{h}}{\partial s}$
Chezy notch	$\dfrac{f}{\beta}\dfrac{u^2}{\sqrt{A}}$	$\propto \dfrac{g}{\sqrt{A}}\dfrac{\partial A}{\partial s}$
Chezy canal	$wf\dfrac{u^2}{A}$	$\dfrac{g}{w}\dfrac{\partial A}{\partial s}$
Mannings' notch	$\dfrac{gn^2}{\beta^{4/3}}\dfrac{u^2}{A^{2/3}}$	$\propto \dfrac{g}{\sqrt{A}}\dfrac{\partial A}{\partial s}$
Mannings' canal	$gwn^2\dfrac{u^2}{A^{4/3}}$	$\dfrac{g}{w}\dfrac{\partial A}{\partial s}$

Therefore the above table shows that we can write the general relationships

$$\frac{\ell\tau}{\rho A} = \frac{bu^2}{A^\lambda}, \quad \& \quad g\frac{\partial \bar{h}}{\partial x} = D(A)\frac{\partial A}{\partial x}$$

and the St Venant equations become

$$\begin{aligned} &A_t + (Au)_x = 0 \\ &u_t + uu_x = gS - \frac{bu^2}{A^\lambda} - D(A)A_x \ . \end{aligned} \tag{36}$$

Non-dimensonalization

This is a process which permits comparing the size of terms in an equation in a meaningful way, ie to determine which terms are small, or large and therefore which dominate the behaviour of the solution. The real art in nondimensionalization is in the choice of scales which are used to make each term dimensionless. This is done by balancing the terms in the equation in a self consistent manner. There is no unique choice, but a properly scaled equations are where the largest dimensional parameters are numerically of order one. However this is not always possible and it is then preferable to try and choose the largest dimensionless parameter to be equal to one, which can only usually be done in a self consistent manner, but requires quite a bit of trial and error. For a more detailed discussion with examples, see Chapter 2 of Fowler, A.C. 1998. Mathematical models in the applied sciences. Cambridge texts in Appl. Maths. Cambridge Univ Press,

We start by defining dimensionless variables (starred quantities) as

$$A^* = \frac{A}{A_o} \qquad u^* = \frac{u}{u_o} \qquad t^* = \frac{t}{t_o} \qquad x^* = \frac{x}{x_o}$$

where A_o, u_o, t_o and x_o are scaling parameters whose values are yet to be determined. For a Chezy canal $D(A) = g/w,\ b = wf,\ \lambda = 1$ and (36) in dimensionless form becomes

$$\begin{aligned} &\frac{A_o}{t_o} A^*_{t^*} + \frac{A_o u_o}{x_o}(A^* u^*)_{x^*} = 0 \\ &\frac{u_o}{t_o} u^*_{t^*} + \frac{u_o^2}{x_o} u^* u^*_{x^*} = gS - \frac{fwu_o^2}{A_o}\frac{(u^*)^2}{A^*} - \frac{g}{w}\frac{A_o}{x_o} A^*_{x^*} \quad . \end{aligned}$$

For cases where advection by the fluid is the most important, then the time scale is given by the advection time scale, i.e.,

$$t_o = \frac{x_o}{u_o},$$

thus after substituting and dropping the '*'s

$$\begin{aligned} &A_t + (Au)_x = 0 \\ &\frac{u_o^2}{x_o}(u_t + uu_x) = gS - \left(\frac{bu_o^2}{A_o}\right)\frac{u^2}{A} - \left(\frac{gA_o}{wx_o}\right)A_x \quad . \end{aligned} \qquad (37)$$

The source and sink terms (gravity, friction, hydrostatic pressure) tend to dominate the flow, hence by convention we balance these terms on the right hand side. For river flow it is sensible to consider a typical volumetric discharge Q_o (tends to be uniform with distance except under flood conditions), then a typical velocity scale is

$$u_o = \frac{Q_o}{A_o}.$$

From the r.h.s. of the momentum eqn;

Balancing 1st and 2nd terms: $gS : \dfrac{fwQ_o^2}{A_o^3} \quad \Rightarrow \quad A_o = \left(\dfrac{fwQ_o^2}{S}\right)^{1/3}$

Balancing 1st and 3rd terms: $gS : \dfrac{gA_o}{wx_o} \quad \Rightarrow \quad x_o = \dfrac{A_o}{wS} = \dfrac{h_o}{S} \quad (A_o = wh_o)$

then the momentum equation now becomes

$$\frac{u_o^2}{gSx_o}(u_t + uu_x) = 1 - \frac{u^2}{A} - A_x \quad ,$$

and since $Sx_o = h_o$ and the Froude number F defined by $F = u_o / \sqrt{gh_o}$, the non-dimensional St Venant equations are

$$\begin{aligned} &A_t + (Au)_x = 0 \\ &F^2(u_t + uu_x) = 1 - \frac{u^2}{A} - A_x \quad . \end{aligned} \tag{38}$$

Example: typical river $Q_o = 30$ m^3/s, $w = 20$ m, $f = 0.05$, $S_0 = 10^{-3}$ giving scale factor values

$$\begin{aligned} A_o &= \left(\frac{fwQ_o^2}{gS}\right)^{1/3} = 45 \text{ m}^2 \\ u_o &= \frac{Q_o}{A_o} = 0.7 \text{ m/s} \\ x_o &= \frac{A_o}{wS} = 2.5 \text{ km} \\ t_o &= \frac{x_o}{u_o} = 3000\, s : \ 1 \text{ hr} \\ h_o &= Sx_o = 2.5 \text{ m} \end{aligned}$$

and Froude number

$$F = \frac{u_o}{\sqrt{gh_o}} = 0.15.$$

Also note that from these scaling we can also write F as

$$F = \left(\frac{u_o^2}{gh_o}\right)^{1/2} = \left(\frac{Q_o^2}{gh_oA_o^2}\right)^{1/2} = \left(\frac{S}{f}\right)^{1/2}$$

and therefore F depends on the roughness and slope of the river.

As we noted earlier, the scaling is not unique and if we were to consider flow on the catchment scale (order 100 km), then the 2 km length scale that came out of the preceding non-dimensionalization is not appropriate. Nor would it be if we were interested in local dynamics such as meanders, dunes and bars which are of the order of 100 m.

Long Wave Theory – Watershed Scale

Returning to the momentum equation in (37) and keeping the balance between the 1st and 2nd terms leaves

$$\frac{u_o^2}{x_o}(u_t + uu_x) = gS\left(1 - \frac{u^2}{A}\right) - \left(\frac{gh_o}{x_o}\right)A_x \ , \qquad A_o = \left(\frac{fwQ_o^2}{gS}\right)^{1/3} .$$

Since interested in flow at the watershed scale O(100 km) then choose ($L \approx 100$ km)

$$x_o = L, \quad \text{with} \quad t_o = \frac{L}{u_o}$$

Then dividing through by gS_0 gives

$$\varepsilon F^2 (u_t + uu_x) = 1 - \frac{u^2}{A} - \varepsilon A_x \tag{39}$$

,
where

$$\varepsilon = \frac{h_o}{LS} .$$

Taking the same scale values as previously, $h_o = 2.5,\ L = 10^5,\ S = 10^{-3}$ gives $\varepsilon = 0.025$, thus for $\varepsilon, F \to 0$ the momentum equation reduces to

$$1 - \frac{u^2}{A} = 0, \quad \text{or} \quad u = \sqrt{A} \tag{40}$$

and mass conservation becomes ($uA = A^{3/2}$)

$$A_t + (A^{3/2})_x = 0 \tag{41}$$

which is the slowly varying flow approximation we considered earlier. This is also known as the kinematic wave approximation to the St Venant equations. Keeping the next order term in the momentum equation gives

$$1 - \frac{u^2}{A} - \varepsilon A_x = 0, \quad \text{or} \quad u = A^{1/2} - \frac{\varepsilon}{2} A^{1/2} A_x + \ldots$$

so mass conservation is now

$$A_t + (A^{3/2})_x = \frac{\varepsilon}{2}(A^{3/2} A_x)_x \tag{42}$$

which is known as the diffusive wave approximation due to the presence of the diffusive term on the right hand side.

Short wave theory

.
Here much shorter length and time scales are important and in particular dynamically generated waves are considered. The starting point is still the momentum equation in (37) with the balance between the 1st and 2nd terms kept

$$\frac{u_o^2}{x_o}(u_t + uu_x) = gS\left(1 - \frac{u^2}{A}\right) - \left(\frac{gh_o}{x_o}\right)A_x \quad .$$

Dividing through by gh/x_o gives

$$F^2(u_t + uu_x) = \delta\left(1 - \frac{u^2}{A}\right) - A_x,$$

where

$$\delta = \frac{x_o S}{h_o}.$$

For small length scales x_o, δ will be small, thus as $\delta \to 0$ the shallow water equations of fluid dynamics are obtained:

$$\begin{aligned} A_t + (uA)_x &= 0 \\ F^2(u_t + uu_x) &= -A_x \quad . \end{aligned} \qquad (43)$$

Waves and Instability

Consider the propagation of a disturbance on a uniformly flowing stream. In particular we want to know what are conditions to cause these disturbances to grow unstably.

Let's take the system

$$\begin{aligned} A_t + (Au)_x &= 0 \\ F^2(u_t + uu_x) &= 1 - \frac{u^2}{A} - A_x \quad . \end{aligned} \qquad (44)$$

At the uniform steady state $u = \sqrt{A}$, but since u and A have been scaled by appropriate values, then u and A are $O(1)$, therefore $u = A = 1$.

Linear Waves:

For small disturbances let $u = 1 + v$ and $A = 1 + a$, substitute into (44) and keep only the dominant highest order terms results in

$$a_t + a_x + v_x = 0, \qquad (45)$$

for mass conservation and

$$F^2(v_t + v_x) = -2v + a - a_x$$
or
$$F^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)v = -2v + a - a_x,$$

for the momentum equation,

Appling the operator $(\partial/\partial t + \partial/\partial x)$ a second time

$$F^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 v = -2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)v + \underbrace{(a_t + a_x)}_{-v_x} - \underbrace{(a_{tx} + a_{xx})}_{-v_{xx}}$$

and rearranging (45) for v_x, results in

$$F^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 v = -2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)v - v_x + v_{xx}. \tag{46}$$

We now look for solutions to (46) of the form

$$v = \exp\left(\frac{ikx + \sigma t}{F^2}\right)$$

which requires

$$\frac{1}{F^2}(\sigma + ik)^2 + \frac{2}{F^2}(\sigma + ik) + \frac{ik}{F^2} + \frac{k^2}{F^4} = 0$$

$$\Rightarrow \ \sigma = -ik - 1 \pm \left(1 - ik - \frac{k^2}{F^2}\right)^{1/2} \quad .$$

An instability will therefore arise if the real part of σ is positive, thus let $\sigma = \sigma_R + i\sigma_I$ and then

$$v = \underbrace{\exp\left(\frac{\sigma_R}{F^2}t\right)}_{\substack{growth \\ rate}} * \exp\left[\underbrace{\frac{ik}{F^2}}_{\substack{wave \\ number}} \ (x + \underbrace{\frac{\sigma_I}{k}}_{\substack{-wave \\ speed}} \ t)\right] \quad .$$

Let the root in σ be given by $p + ikq = \left(1 - ik - \dfrac{k^2}{F^2}\right)^{1/2}$ thus

$$\sigma = -ik - 1 \pm (p + ikq)$$

and

$$\sigma_R = -1 \pm p, \quad \sigma_I = -k(1 \pm q). \tag{47}$$

For an instability to grow we must have $\sigma_R > 0$, or from (47), $p > 1$.

From the definition of $p + iq$

$$p^2 - k^2q^2 + 2kqpi = 1 - \frac{k^2}{F^2} - ik \ ,$$

thus

$$p^2 - k^2q^2 = 1 - \frac{k^2}{F^2}, \qquad 2qp = -1 \ \Rightarrow \ q = \frac{-1}{2p}, \tag{48}$$

or

$$p^2 - \frac{k^2}{4p^2} = 1 - \frac{k^2}{F^2}.$$

Define the operator $L(p)$ as

$$\begin{aligned} L(p) &= p^2 - \frac{k^2}{4p^2} \\ &= 1 - \frac{k^2}{F^2} \end{aligned}.$$

$L(p)$ is an increasing function of p $[L'(p) = 2p + k^2/2p^3]$ for $p > 0$, thus to have solutions that satisfy $p > 1$, then need $L(p) > L(1)$ (since as p goes through 1, L must be increasing).

For any p, $L(p) = 1 - \frac{k^2}{F^2}$ and specifically $L(1) = 1 - \frac{k^2}{4}$, thus we require ($L(p) > L(1)$)

$$1 - \frac{k^2}{F^2} > 1 - \frac{k^2}{4} \qquad \Rightarrow \ F > 2.$$

(a) $F > 2$, Roll waves or Verdernikov instability

For this case $q = -1/2p < 0$ since $p > 1$. The wave speed is given by

$$-\frac{\sigma_I}{k} = 1 \pm q = 1 \pm \frac{1}{2p} \quad (\frac{3}{2} \text{ or } \frac{1}{2} \text{ for } \ p = 1),$$

hence there are two waves and both travel downstream. From (47) one of the waves where $\sigma_R = -(1+p) < 0$ is stable while the other wave for which $\sigma_R = -1 + p > 0$ is unstable.

(b) $F < 2$, i.e. $p < 1$

The speeds for both waves are

$$-\frac{\sigma_I}{k} = 1 \pm q = 1 \pm \frac{1}{2p},$$

are always positive and therefore flow downstream provided $p > 1/2$. Note that there will always be a wave that travels downstream for $p > 0$ corresponding to

$$-\frac{\sigma_I}{k} = 1 + \frac{1}{2p}.$$

For a wave to travel upstream we require

$$-\frac{\sigma_I}{k} = 1 - \frac{1}{2p} < 0,$$

which gives $p < 1/2$. Therefore using the same argument as earlier, we want $L(p) < L(1/2)$ or

$$1 - \frac{k^2}{F^2} < \frac{1}{4} - k^2$$
$$\Rightarrow \quad F < \left[\frac{k^2}{3/4 + k^2} \right]^{1/2}.$$

As F depends on k, then the above condition is satisfied for $0 < F < 1$.

Both waves are stable (as $F < 2$) with one wave travelling downstream given by $-\sigma_I / k = 1 + 1/(2p) > 0$ and one travelling upstream given by $-\sigma_I / k = 1 - 1/(2p) < 0$ for $p < 1/2$.

\
(c) $1 < F < 2$, i.e. $1/2 < p < 1$

Since $p > 1/2$ then the wave $-\sigma_I / k = 1 - 1/(2p) > 0$ goes downstream as well now. Thus both waves again travel downstream and both are stable (as $F < 2$).

Subcritical flow is defined where $F < 1$ and since waves can travel in both directions, any disturbance can propagate and be felt upstream. Solutions to the St Venant equations therefore require a boundary condition both upstream and downstream.

Supercritical flow is defined where $F > 1$ for which disturbances only travel downstream. Thus the flow at any time is totally unaware of anything that is happening ahead of it and solutions of the St Venant equations then require both boundary conditions to be specified upstream.

Sediment Transport

Sediment is generally classified by particle size with the following types:

- clay - diameter < 2 μm
- silt - (2 – 60) μm
- sand – 60 μm – 1 mm
- gravel - > 1 mm
- shingle, cobbles and boulders

Clay particles are cohesive and possess a surface charge which results in the preferential sorption of chemical contaminants (e.g. nitrogen phosphorous, pesticides, herbicides, insecticides) and microbial pathogens (e.g. bacteria, viruses). On the other hand sands particles and above are non-cohesive.

Sediments are transport either in suspension (known as suspended load) or as bedload, and when transported in both forms then it is known as the total load.

Several definitions of sediment size are used since sediments are not usually spherical.

a) The sieve diameter : gives the size of particle that passes through a square mesh sieve of a given size but not through the next smallest sieve size, i.e. 1 mm < d_s < 2 mm

b) Sedimentation character: the size of a quartz sphere that has the same settling velocity in the fluid as the real sediment particle.

c) Nominal diameter: size of sphere of the same density and mass as the actual particle.

There is no direct way of determining shape, however a shape factor is often used and it is usual to assume

$$\text{Shape factor} = \frac{\text{surface area of particle}}{\text{surface area of sphere of same volume}} ,$$

however a particle could be round or cubic in shape, the shape factor for a cubic particle is 0.806.

Density of particles: quartz and clay minerals have a density of ρ_s = 2650 kg/m^3, natural sediments have densities similar to that of quartz.

Relative density or specific gravity γ_s is defined as $\frac{\rho_s}{\rho}$, which gives 2.65 for water and 2200 for air. Sand particles can be carried by wind, sand storms in Sahara, movement of sand dunes across the desert can cause towns to buried (Timbuktu)

Particle Size Distribution

Natural sediments are mixtures of many different particle sizes and shapes. The particle size distribution is usually given by a plot of the cumulative fraction of particles against particle size.

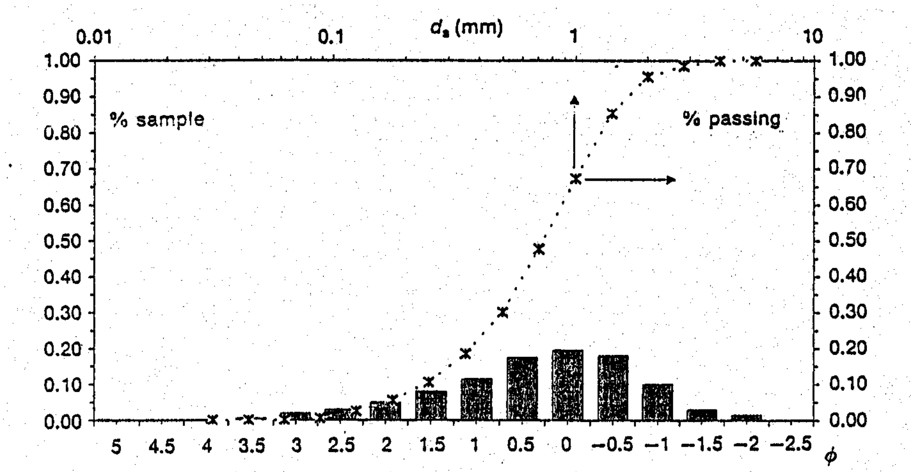


Figure 1. Typical particle size distribution curve. (a) Percentage sampling as a function of sedimentological size parameter φ (linear scale). (b) Cumulative percentage passing as a function of the particle size d_s in mm (semilog scale).

The characteristic sediment size which is used for most transport studies is the median grain size, ie d_{50} and defines the sediment size by which 50% by weight of all sediments is finer (or larger).

You can also talk about characteristic sizes d_{10}, d_{75}, d_{90} which refer to particle sizes where 10%, 75% & 90% of all the sediment is finer than, respectively.

Range of particle sizes can also be expressed through a sorting coefficient S_{ort} defined as

$$S_{\mathrm{ort}} = \sqrt{\frac{d_{90}}{d_{10}}},$$

so that S_{ort} small (near one) implies a nearly uniform sediment size distribution, while S_{ort} large implies a broad size distribution.

Particle Fall velocity

When particles are in suspension in a fluid layer, then they fall back towards the bed due to gravity. The particle will initially accelerate towards a limiting fall velocity w_o at which the immersed weight of the particle becomes balanced to its drag force.

$$\text{Immersed weight} = \text{mass x g} = \text{density x Volume x g.}$$
$$= (\rho_s - \rho)\, Vg$$

$$\text{Drag force} \quad F_D = \frac{1}{2} C_D \rho A v_s^2$$

where v_s = particle fall velocity, A = cross-sectional area of particle, V = particle volume, g = gravity and C_D = drag coefficient. The fall velocity v_s is then defined by

$$(\rho_s - \rho) V g = \frac{C_D}{2} \rho A v_s^2 .$$

For a spherical particle $V = \frac{4}{3}\pi r^3 = \frac{\pi d^3}{6}$ and $A = \pi r^2 = \frac{\pi d^2}{4}$, therefore

$$(\gamma_s - 1)\frac{\pi d^3}{6} g = \frac{C_D}{2}\frac{\pi d^2}{4} v_s^2 ,$$

giving the fall velocity

$$v_s = \sqrt{\frac{4}{3}\frac{gd}{C_D}(\gamma_s - 1)} .$$

Figure 2 below is a plot the drag coefficient C_D as a function of the Reynolds number Re where $\mathrm{Re} = \frac{v_s d_s}{\nu}$, ν is the kinematic viscosity.

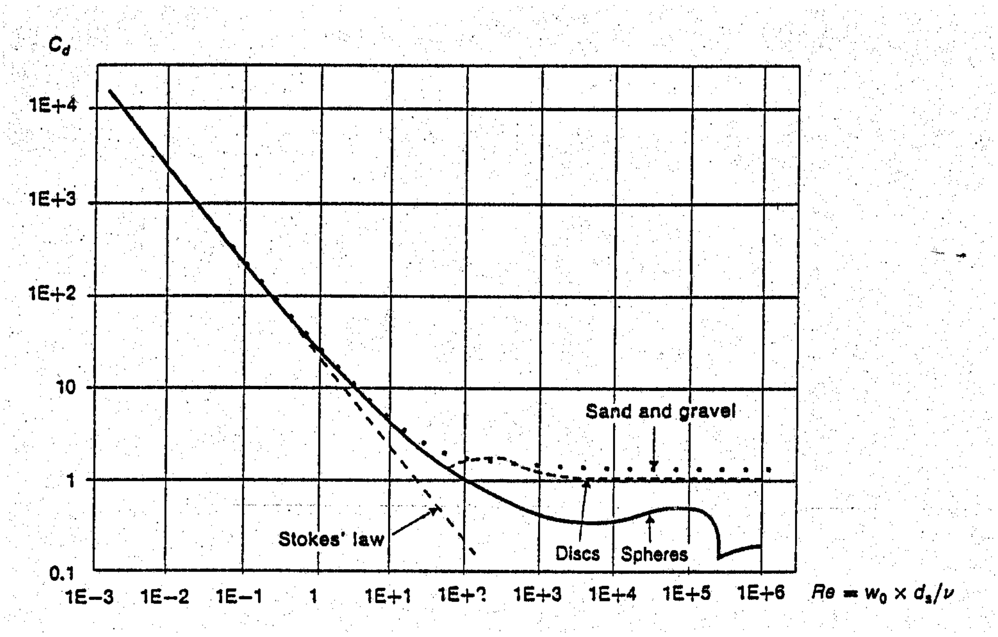


Figure 2. Drag coefficient of a single particle in a still fluid

For laminar flow around spherical particles, Re < 1 and C_D is given by $C_D = \frac{24}{Re}$. For turbulent flow around spherical particles Re > 1000 and $C_D \approx$ constant. For natural particles which have irregular shapes, C_D varies from that of spherical particles, and as in fact greater. For sands and gravels a simple approximation is:

$$C_D = \frac{24}{Re} + 1.5 \qquad \text{Re} < 1x10^4$$

Initiation of motion

The basic mechanism responsible for sediment transport is the drag force exerted by the fluid on the individual grains. The cumulative effect of all drag forces is also opposed by the bed shear stress τ_b which acts on the flow to give non-uniform flow velocities.

As discharge or flow velocity increases, the bed shear stress will also increase according to

$$\tau_b = \rho g R S$$

where R = hydraulic radius and S = bed slope. Note as flow velocity increases, so does R due to the increased flow depth.

Once the shear stress approaches the value known as the critical shear stress $\tau_{c,}$ particles on the bed will begin to move. This is known as the threshold condition as for $\tau_b < \tau_{c,}$, there is no bed movement.

Sediment transported tends to be characterized by two types, there are

(a) Bed load: where the drag force is dominant with little lift force and particles just roll, slide or saltate along the bed, and

(b) Suspended load: where there is a large lift force due to turbulence and particles are lifted from the bed are carried in suspension with the main flow.

Bed Formation

With progressive increases in flow velocity we begin to see changes in the bed formation as follows:

(i) At low flow velocity the bed does not move, $\tau_b < \tau_c$ and threshold conditions not yet reached

(ii) When the velocity is near the threshold, $\tau_b \sim \tau_c$, the bed begins to move

(iii) A further increase in velocity causes ripples like a saw-tooth section to appear

(iv) At higher velocity again, large periodic irregular structures occur, ie dunes with ripples superimposed

(v) At still higher velocities ripples disappear and give way to large dunes

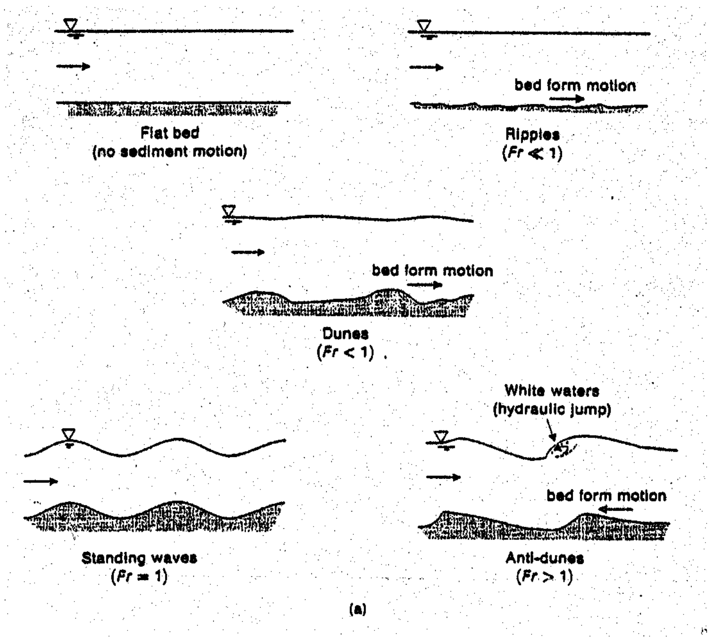


Figure 3. Typical bed forms

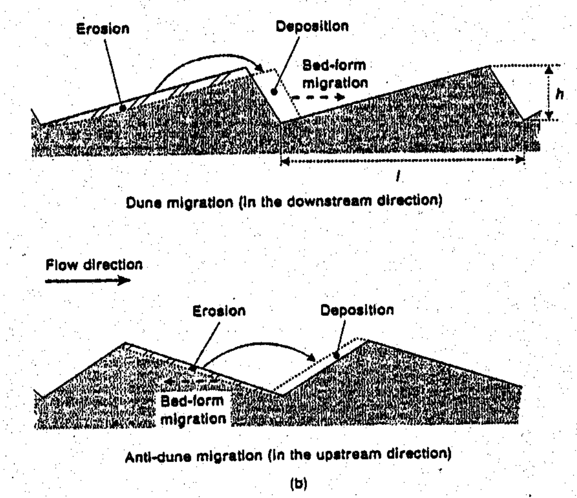
Both the ripples and dunes slowly migrate downstream through the scouring of material from the upstream dune face and its subsequent deposition on the downstream face. See top diagram of Figure 6.

(vi) the next stage is that the dunes are washed out by the flow and leaving again a flat bed, for Froude number (F_r) close to 1

(vii) for a further increase in velocity such that $F_r > 1$, you get standing waves since the sand waves (dunes) are in phase with the surface water waves

(viii) For $F_r >> 1$, surface waves become steep and break. There is also a gradual movement upstream of the sand dunes – called anti-dunes

Anti-dunes slowly migrate upstream through the scouring of material from the downstream face of one dune and its deposition on the upstream face of the next dune. See bottom diagram of Figure 6.



Photograph 4 Sediment ripple formation.

Figure 4. Bed form motions

Bed form (1)	Flow (2)	Bed form motion (3)	Comments (4)
Flat bed	No Flow (or $Fr \ll 1$)	NO	No sediment motion.
Ripples	$Fr \ll 1$	D/S	Three-dimensional forms. Observed also with air flows (e.g. sand ripples in a beach caused by wind).
Dunes	$Fr < 1$	D/S	Three-dimensional forms. Sand dunes can also be caused by wind.
Flat bed	$Fr \leq 1$	NO	Observed also with wind flow.
Standing waves	$Fr = 1$	NO	Critical flow conditions. Bed standing waves in phase with free-surface standing waves.
Antidunes	$Fr > 1$	U/S	Supercritical flow with tumbling flow and hydraulic jump upstream of antidune crests.
Chute-pools	$Fr > 1$	U/S	Very active antidunes.
Step-pools	$Fr > 1$	–	Cascade of steps and pools. Steps are often caused by rock bed.

References: Henderson (1966), Graf (1971).
Notes: D/S = in downstream flow direction; Fr = Froude number; U/S = in upstream flow direction.

Analytical approach to the Threshold of Motion

If looked closely at an erodible bed, it is clear that some particles are more exposed to fluid flow than others and are therefore more prone to move as shown in Figure 4

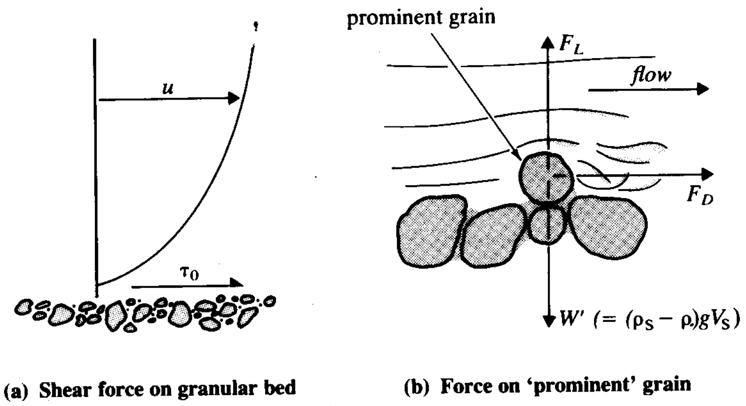


Figure 5. Fluid forces causing sediment movement

The external forces acting on these particles are:

- the flow pattern or drag
- submerged weight and angle of repose.

The number of prominent particles in a given surface area is related to the areal grain packing defined as

$$\frac{\text{area of grains}}{\text{total area}} = \text{area of influence}$$

For a spherical particle the area of influence must be proportional to d^2 where d diameter of particle. Thus the drag force will be given by (pressure x area of influence)

$$F_D = \frac{\tau_b d^2}{a}$$

where a = constant of proportionality. At the threshold of movement, F_D must be balanced by the friction force $F_R = \mu N$ where

N = normal force

= submerged weight

$= (\rho_s - \rho)Vg$

$= (\rho_s - \rho)\dfrac{\pi d^3}{6} g$ for a spherical particle

therefore (with $\mu = \tan\phi$)

$$F_R = (\rho_s - \rho)\frac{\pi d^3}{6} g \tan\phi .$$

Finally equating the expressions for F_D and F_R and rearranging gives the dimensionless equation

$$\frac{\tau_b}{(\rho_s - \rho) g d} = \frac{\pi a \tan\phi}{6} \qquad (49)$$

This equation is only valid if the force on each particle acts through the centre of gravity, and that lift due to turbulent fluctuations acting on the bed can be ignored.

Shields carried out a series of experiments and related particle movement to Reynolds number, more particularly to the grain Reynolds number, as particle movement should relate to conditions occurring at the particle rather than in the general fluid flow.

Grain Reynolds number Re* is given by

$$\mathrm{Re}^* = \frac{u^* d}{\nu}$$

where $u^* = \sqrt{\frac{\tau_b}{\rho}}$, is the friction velocity and ν is the kinematic viscosity.

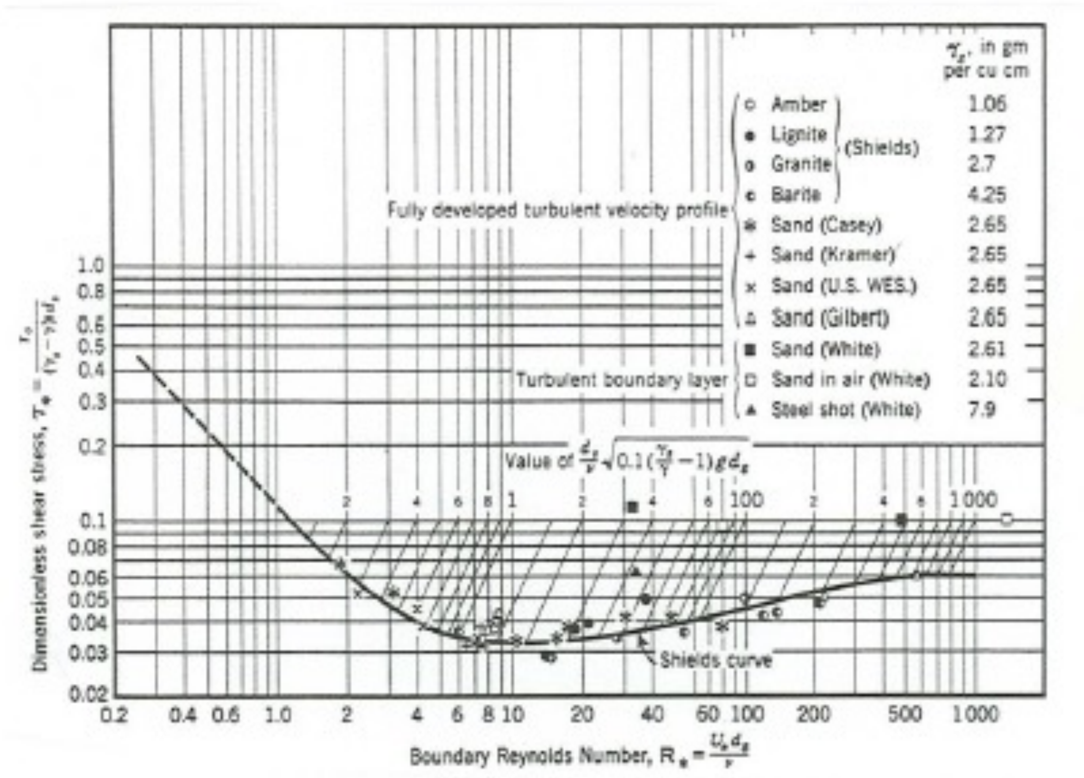


Figure 6. Shields' Diagram plot of $\frac{\tau_b}{(\rho_s - \rho) g d}$ against Re*

Figure 6 shows a plot of $\dfrac{\tau_b}{(\rho_s - \rho)\, gd}$ against Re* and is referred to as the Shields diagram. Above the marked line particles are in motion, while below the line there is no motion.

For $Re^* > 400$, the threshold line is constant at 0.056, therefore the critical shear stress for movement for this region is given by

$$\frac{\tau_{cr}}{(\rho_s - \rho)\, gd} = 0.056$$

Substituting for the $\tau_{cr} = \rho g R S_o$

$$\frac{\rho g R S}{(\rho_s - \rho)\, gd} = 0.056 \qquad (50)$$

gives an equation for finding the minimum particle size that is stable (ie not moving) for a given channel design (defined through S_o and R)

Bedload Transport

There are various empirical formulas for bedload transport, one of the more popular is the Meyer-Peter-Muller formula, which in dimensionless form is

$$q_b^* = K(\tau^* - \tau_{cr}^*)^{3/2} \qquad (51)$$

,
where

$$q_b^* = \frac{q_b}{[(\rho_s - \rho) g d^3 / \rho]^{1/2}}, \quad \tau^* = \frac{\tau_b}{(\rho_s - \rho) g d}$$

and

q_b = bedload transport rate (Vol flow/unit width/time = m^2/s), τ^* = Shield's stress, τ_{cr}^* = dimensionless critical shear stress = 0.06, d = median grain size and $K : 8$.

Exner Equation

Take an (x,z) coordinate geometry with x in the downstream direction and parallel with the average bed slope, and z = height perpendicular to the average bed slope. Let $z = \eta$ = water surface and $z = s$ = bed height, hence the water flow depth $h = \eta - s$. To determine the mass conservation for the bed height, again consider a control volume (CV) of length Δx. The mass of sediment within the CV is

mass = $(1-n)\rho_s s w \Delta x$ where w = flow width and n = bed porosity (0.3 – 0.4)

bedload flux = $\rho_s w q_b$

and therefore mass conservation is given by

$$(1-n)\frac{\partial s}{\partial t}+\frac{\partial q_b}{\partial x}=0. \tag{52}$$

To examine the qualitative behaviour of this equation, consider very low Froude number flows, i.e. $F \ll 1$. Thus we can assume that there is negligible movement of the water surface, i.e. the height of the water surface is constant. Therefore $\eta = h_o$, and $h = h_o - s$. Note that the $z = 0$ datum can be aligned on the average bed slope. Since $q = uh$ = water flux, then we can write the bed shear stress as

$$\tau_b = f\rho u^2 = \frac{f\rho q^2}{(h_o - s)^2},$$

and the bedload flux as

$$q_b(\tau_b) = q_b\left[\frac{f\rho q^2}{(h_o - s)^2}\right] = (1-n)\tilde{q}_b(s)$$

with $\tilde{q}_b(s)$ being an increasing function of s. Substitution into mass conservation gives

$$\frac{\partial s}{\partial t}+\frac{\partial \tilde{q}_b}{\partial x}=0$$

or

$$\frac{\partial s}{\partial t}+\tilde{q}_b'(s)\frac{\partial s}{\partial x}=0.$$

From the method of characteristics we have

$$\frac{ds}{dt}=0 \quad \text{on} \quad \frac{dx}{dt}=\tilde{q}_b'(s)$$

thus:

(i) if s is small as in a perturbation then an instability cannot grow in time as s = constant on a characteristic. Therefore the bed is neutrally stable.

(ii) qualitatively we expect $\tilde{q}_b'(s)$ being an increasing function of s which implies a higher local bed height will have a greater characteristic speed. Thus any mound shaped initial profile will lead to shock formation which then travels downstream.

Overall then a bedload transport model on its own is not sufficient to develop instabilities resulting in dune or anti-dune migration. Let us next consider a suspended sediment transport model.

Suspended Sediment Transport

In this model we will assume that bedload transport can be neglected (q_b = 0). Consequently changes in the bed height of the river occur as a result of the entrainment of particles into the flow which are later deposited, due to gravity, back onto the bed some distance downstream.

Define c = suspended sediment concentration (kg/m^3), then mass conservation for c is given by (sed. mass in CV = $hcw\Delta x$, susp. sed. flux = qcw)

$$\frac{\partial(hc)}{\partial t}+\frac{\partial(qc)}{\partial x}=\rho_s(\vartheta_E-\vartheta_D), \tag{53}$$

where

ϑ_E = sediment entrainment (erosion) rate (m/s)

ϑ_D = sediment deposition (gravity) rate (m/s); $\rho_s\vartheta_D=v_s c$.

A typical functional form for the erosion rate is dependent on the excess bed shear stress and ϑ_E can be written as $\vartheta_E=v_s E$ where

$$E\propto(\tau^*-\tau^*_{cr})^{3/2}\,\mathrm{Re}^{*1/5}.$$

Note E increases with τ and therefore u, and is the order of 10^{-3} – 10^{-1}. Our model equations comprise of combining mass conservation for suspended sediment with the Exner equation modified to also account for the effects of erosion and deposition on the bed height. Thus

$$\begin{aligned}\frac{\partial(hc)}{\partial t}+\frac{\partial(qc)}{\partial x}&=\rho_s(\vartheta_E-\vartheta_D)=v_s(\rho_s E-c)\\(1-n)\frac{\partial s}{\partial t}&=\vartheta_D-\vartheta_E=v_s(\frac{c}{\rho_s}-E)\quad.\end{aligned}\tag{54}$$

Non-dimensionlization

Choosing scales for *c, x, t, h, q* and *s* with subscripts "o", and taking $E = E(u)$ as $\tau(u)$ with E_o being a typical value. The suspended sediment equation becomes (all variables are now dimensionless)

$$\frac{h_o c_o}{t_o}\frac{\partial(hc)}{\partial t}+\frac{q_o c_o}{x_o}\frac{\partial(qc)}{\partial x}=v_s(\rho_s E_o E-c_o c).$$

Balancing the erosion and deposition rates gives the concentration scale: $c_o=\rho_s E_0$

To get a bed form length scale we next balance the spatial derivative coefficient with the right hand side:

$$\frac{q_o c_o}{x_o}=v_s c_o\ \Rightarrow\ x_o=\frac{q_o}{v_s}=\frac{u_o h_o}{v_s}$$

Physically this is saying that the bedform length scale is determined as the ratio of the horizontal flow rate to the vertical fall velocity, ie it is a measure of the downstream distance travelled by a particle in a fluid flux q_0 before being deposited back on the bed again. The conservation equation is now

$$\frac{h_o}{v_s t_o}\frac{\partial(hc)}{\partial t}+\frac{\partial(qc)}{\partial x}=E(u)-c\,.$$

We are still in need of a time scale, however the evolution of a bedform is much slower than the travel timescale of a particle in suspension, hence the time scale needs to be chosen from the bed mass conservation equation. In dimensionless form this is

$$(1-n)\frac{s_o}{t_o}\frac{\partial s}{\partial t}=v_s E_o(c-E(u))\,.$$

Balancing both sides gives the timescale as: $(1-n)\frac{s_o}{t_o}=v_s E_o \;\Rightarrow\; t_o=\frac{(1-n)s_o}{E_o v_s}$

As $h=h_o-s$ then choose $s_o = h_o$, thus

$$t_o=\frac{(1-n)h_o}{E_o v_s}\;\Rightarrow\;\frac{h_o}{v_s t_o}=\frac{E_o}{1-n}=\varepsilon\,.$$

Our dimensionless system is

$$\begin{aligned}\varepsilon\frac{\partial(hc)}{\partial t}+\frac{\partial(qc)}{\partial x}&=-\frac{\partial s}{\partial t}\\ \frac{\partial s}{\partial t}&=c-E(u)\;.\end{aligned}\qquad(55)$$

Typical values: $\rho_s=2650\,\text{kg/m}^3$, $E_0 = 1/300$, $n = 0.4$, $v_s=0.01\,\text{m/s}$ thus $c_o\approx 10\,\text{kg/m}^3$, and

$\varepsilon=1/(0.6*300)\approx 0.005$.

Since $\varepsilon<<1$, c reacts on a fast advective timescale, however the bed evolves on a much slower timescale of

$t_o=\frac{(1-n)h_o}{E_o v_s}\approx 1.8\text{x}10^4 h_o$ secs, > 1 hr for $h_o=0.5\,\text{m}$.

As $\varepsilon\to 0$ our leading order system is

$$\frac{\partial(qc)}{\partial x} = -\frac{\partial s}{\partial t} = E(u) - c = \hat{E}(s) - c \quad , \tag{56}$$

since $u = \dfrac{q}{h_o - s}$ and in non-dimensional form $u = \dfrac{1}{1-s}$.

Linear Stability

Expand (56) around the basic state of $s = 0$, $c = 1$ and $\hat{E}(0) = 1$, thus letting $c = 1 + C$ we have

$$\begin{aligned} \frac{\partial C}{\partial x} &= [\hat{E}(0) + s\hat{E}'(0)] - (1 + C) \\ &= \hat{E}'(0) - C = -\frac{\partial s}{\partial t} \quad . \end{aligned} \tag{57}$$

Let $C = Ae^{\sigma t + ikx}$, $s = Be^{\sigma t + ikx}$ then from (57)

$$Aik = \hat{E}'(0)B - A = -\sigma B \, . \tag{58}$$

Equating first and middle term gives $B = (1 + ik)A / \hat{E}'(0)$ and then from equating the middle and last term results in

$$\sigma = -\frac{\hat{E}'(0)(k^2 + ik)}{1 + k^2} . \tag{59}$$

Since $\hat{E}'(0) > 0$ then $\mathrm{Re}(\sigma) < 0$ and the basic state is stable.

St Venant with suspended sediment and bedload

It has been demonstrated so far that bedload by itself, or suspended sediment with an Exner equation for the bed evolution are not sufficient for an instability to develop and bedforms to grow. This suggests that we may need to include the effect of interactions between the fluid flow and the bed evolution. Thus a more complete model is given by combining mass conservation for water (St Venant equations), suspended sediment and the bed (Exner) resulting in the following system (s = bed elevation, η = water surface elevation)

Water: (Chezy canal)

$$\begin{aligned} &\eta - s = h \\ &h_t + (hu)_x = 0 \\ &u_t + uu_x = g(S - \eta_x) - \frac{fu^2}{h} \quad . \end{aligned} \tag{60}$$

Sediment:

$$\begin{aligned}&\frac{\partial(hc)}{\partial t}+\frac{\partial(uhc)}{\partial x}=\rho_s(\vartheta_E-\vartheta_D)\\&(1-n)\frac{\partial s}{\partial t}+\frac{\partial q_b}{\partial x}=\vartheta_D-\vartheta_E\quad .\end{aligned} \tag{61}$$

As always we want to non-dimensionlize, so take the following scalings

$$c_o=\rho_s E_o,\quad x_o=\frac{q_o}{v_s}=\frac{u_o h_o}{v_s},\quad \varepsilon=\frac{E_o}{1-n},\quad t_o=\frac{(1-n)h_o}{v_s E_o}=\frac{h_o}{\varepsilon v_s}.$$

From the momentum equation (variables now dimensionless)

$$\frac{u_o}{t_o}u_t+\frac{u_o^2}{x_o}uu_x=gS-\frac{g\eta_o}{x_o}\eta_x-f\frac{u_o^2}{h_o}\frac{u^2}{h}\ ,$$

then by balancing the gravity and friction terms, and substituting for t_o and x_o we have

$$\frac{v_s u_o}{h_o}\left(\varepsilon u_t+uu_x\right)=gS\left(1-\frac{u^2}{h}\right)-\frac{g\eta_o v_s}{u_o h_o}\eta_x\ , \tag{62}$$

with

$$h_o=\left(\frac{fq_o^2}{gS}\right)^{1/3}.$$

Choosing $\eta_o=h_o,\ F=\dfrac{u_o}{\sqrt{gh_o}}$, (62) becomes

$$F^2\left(\varepsilon u_t+uu_x\right)=\delta\left(1-\frac{u^2}{h}\right)-\eta_x\ , \tag{63}$$

where

$$\delta=\frac{u_o S}{v_s}=\frac{x_o S}{h_o}.$$

Applying the same scalings to the remaining two equations in (60) along with (61) and $s_o=h_0$ results in

$$\begin{aligned}&\eta-s=h\\&\varepsilon h_t+(hu)_x=0\\&F^2\left(\varepsilon u_t+uu_x\right)=\delta\left(1-\frac{u^2}{h}\right)-\eta_x\\&h(\varepsilon c_t+uc_x)=E(u)-c\\&s_t+\beta(q_b)_x=-\left(E(u)-c\right)\ .\end{aligned} \tag{64}$$

The parameter β in the final equation in (64) is given by { $(q_b)_o$ is the bedload flux scaling}

$$\beta = \frac{(q_b)_o}{\rho_s E_0 q_o} = \frac{(q_b)_o}{q_o c_o} = \frac{\text{bedload flux}}{\text{suspended sed flux}}.$$

Consider flows where the suspended sediment flux dominates, hence $\beta << 1$, so for $\varepsilon, \beta \to 0$ (64) reduces to

$$\begin{aligned}
&\eta - s = h \\
&(hu)_x = 0 \;\Rightarrow\; uh = 1 \\
&F^2 u u_x = \delta\left(1 - \frac{u^2}{h}\right) - \eta_x \qquad\qquad (65) \\
&huc_x = E(u) - c \\
&s_t = -\big(E(u) - c\big)\ .
\end{aligned}$$

For typical values of $u_o = 1\text{m/s},\ v_s = 10^{-2}\,\text{m/s},\ S = 10^{-3} \;\Rightarrow\; \delta = 0.1$, then for $\delta \to 0$ also,

$$\begin{aligned}
&\eta - s = h \\
&uh = 1 \\
&\eta_x + F^2 u u_x = 0 \qquad\qquad (66) \\
&huc_x = E(u) - c \\
&s_t = -\big(E(u) - c\big)\ .
\end{aligned}$$

Integrating the momentum equation (using upstream boundary condition $u = h = \eta = 1,\ s = 0$)

$$\eta + \frac{F^2}{2}u^2 = 1 + \frac{F^2}{2},$$

and since

$$\begin{aligned}
s_t &= \eta_t - h_t \quad (h = 1/u) \\
&= -F^2 u u_t + \frac{u_t}{u^2}\ .
\end{aligned}$$

Finally the last two equations in (65) can now be written as

$$\begin{aligned}
c_x = E(u) - c &= -s_t \\
&= F^2 u u_t - \frac{u_t}{u^2}\ .
\end{aligned} \qquad (67)$$

Stability analysis

Let $u = 1 + U$, $c = 1 + C$ and linearizing around $u = c = 1$, then (67) becomes

$$C_x = UE'(1) - C = (F^2 - 1)U_t\,.$$

Look for solutions of the form $C = Ae^{\sigma t+ikx}, \quad U = Be^{\sigma t+ikx}$

$$Aik = E'(1)B - A = (F^2-1)\sigma B$$

Equating first and middle term gives $A = BE'(1)/(1+ik)$ and then from equating the middle and last term results in

$$\sigma = \frac{E'(1)(k^2+ik)}{\left(F^2-1\right)\left(1+k^2\right)}.$$

The real part of σ is given by

$$\mathrm{Re}(\sigma) = \frac{E'(1)k^2}{\left(F^2-1\right)\left(1+k^2\right)}, \tag{68}$$

and since $E'(1) > 0$, then for $F > 1$ $\mathrm{Re}(\sigma) < 0$. We therefore have an instability resulting in anti-dunes that travel upstream as the wave speed

$$-\frac{\mathrm{Im}(\sigma)}{k} = \frac{-E'(1)}{\left(F^2-1\right)\left(1+k^2\right)} < 0,$$

for $F > 1$. Lastly note that this is still not well posed as (68) shows that the model is unstable for arbitrary small wavelengths.