O1 History of Mathematics Lecture XIII Complex analysis

Monday 19th November 2018 (Week 7)

## Summary

Complex numbers: validity and representation

Substitution of complex values for real

Cauchy's contributions

Riemann

▶ What *is* an analytic function?

## Early ideas about complex numbers

Before 1600, very faint beginnings:

Cardano (1545) [from quadratics]

Bombelli (1572) [from cubics]

Harriot (c. 1600) [from quartics]

But:

For the most part such roots were ignored: negative roots were described merely as 'false', but complex roots as 'impossible'. (Mathematics emerging, p. 459.)

### Cardano and complex numbers

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tamë huic quantitati,que uere eft fophiftica, quoniam per eam, non ut in puro m: nec in 25 m:m: 15 qd.eft

Q.v.s.s.tio 1111. Fac de 6 duas partes, quarum quadrata iméta fins 50, hæc folui tur per primam, non per focundam regulam, eft enim de puro m: ideo due 3 dimidium 6 in fe, fir 9, minute ex dimidio 50, quod eft 25, fir res due 3 dimidium 6 in fe, fir 9, minute ex dimidio 50, quod eft 25, fir res Problem: find two numbers that add to 10 and multiply to 40, i.e., solve an equation of the type 'square plus number equals thing'

Cardano noted that  $5 + \sqrt{-15}$  and  $5 - \sqrt{-15}$  solve the problem, "dismissis incruciationibus", meaning

"putting aside mental tortures", or

"the cross-multiples having canceled out",

or

"the imaginary part being lost"

But regarded such ideas as absurd and useless

## Bombelli and complex numbers

PRIMO. 169 Ho trouato un'altra forte di R.c.legate molto differen ri dall'altre, laqual nafce dal Capitolo di cubo eguale à tanti e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della meta del numero come in effo Capitolo fi dimostrarà, laqual forte di z. q. hànel fuo Algorifmo diuerfa operatione dall'altre, e diuerfo nome ; per che quando il cubato del terzo del li tànti è maggiore del quadrato della metà del numero: lo ecceffo loro non fi può chiamare ne più ne meno però lo chiamarò più di meno, quando celi fi doue rà aggiongere, e quando fi douerà cauare, lo chiamerò men di meno, e questa operatione è necessariissima più che l'altre R.c. L.per rifpetto delli Capitoli di potenze di potéze, accompagnati có li cubi, ò tanti, ô con tutti due infieme, che molto più fonoli cafi dell'agguagliare doue ne nafce quefta forte di R. che quelli doue nasce l'altra, la quale parerà à molti più tofto fofiftica, che reale, e tale opinione hò tenuto anch'io, fin' che hò trouato la fua dimoitratione in lince ( come fi dimostrarà nella dimostratione del detto Capitolo in superficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

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Più uia più di meno, fa più di meno. Meno uia più di meno, fa meno di meno. Più uia meno di meno, fa meno di meno. Meno uia meno di meno, fa più di meno. Più di meno uia più di meno, fa più. Meno di meno uia più di meno, fa più. Meno di meno uia men di meno fa più.

Si

"Another sort of cube root much different from the former ..."

### Systematic rules:

più di meno via più di meno, fà meno  $(\sqrt{-1} \times \sqrt{-1} = -1)$ meno di meno via più di meno, fà più  $(-\sqrt{-1} \times \sqrt{-1} = 1)$ 

But complex numbers were not admitted as solutions of equations — they could appear in calculations, provided they cancelled out by the end

Complex numbers justified through practical use?

## Harriot and complex numbers

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### Add MS 6783 f. 156

Unpublished manuscripts contain systematic treatment of complex roots of equations — but these were removed by his editors

Cf. Harriot's *Artis analyticae praxis* (1931), pp. 14–15; see:

Muriel Seltman & Robert Goulding, Thomas Harriot's Artis analyticae praxis: an English translation with commentary, Springer, 2007

### Descartes and 'imaginaries'

380 LA GEOMETRIE, choicnt 5, 1, 8, 5, 8 que celles de la premiere effoient  $\frac{3}{2}V_{3}, \frac{1}{2}V_{3}$ ,  $\frac{1}{2}V_{3}$ . Cancon Carecoporation peut aufly feruir pour rendre la quanmente tité connuë de quelqu'un des termes de l'Equatiõe figale conta a quelque autre donnée, comme fia yant terme  $\frac{1}{2}V_{1} - \frac{1}{2}V_{2} + \frac{1}{2}V_{2}$ . Serve  $\frac{1}{2}V_{1} - \frac{1}{2}V_{2} + \frac{1}{2}V_{2}$ .

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Au refle rant les vrayes racines que les faufles ne four marrar pas toufiours recelles, mais quelquelos feulement imagine autor quelquelos acuton peut bienconfours en imagine marrar quelquelos acutone quantité, qui correlponde a celles menter quelquelos acutone quantité, qui correlponde a celles nerroiseti celle cy', s'  $-6\pi^2 + 13\pi - 1030$ , il ny en a routerois qu'or necle, qui cit a, & pour les deux autres, quoy qu'on les agmente, ou diminae, ou multiplie en la fiçon que le viens d'expliquer, on ne foauroit les rendireattres qu'in agminares.

Livés-Or quand pour trouver la confraction de quelque grantés probleme, on vient avne Equation, en laquelle la quande de la contra de la confraction de la confractione la confractione a trois dimensions, en la quelle la quanproblement de la confractione de la confracne de la confractione de la confractione de la confracne de la confractione de la confractione de la confracne de la confractione de la confractione de la confracne de la confractione de la confractione de la confractione de la confracne de la confractione de la confractione de la confractione de la confracne de la confractione de la confra

### La géométrie (1637):

introduced the term 'imaginaire' — meant to be derogatory?

Didn't regard them as numbers

## Ideas about complex numbers in the later 17th century

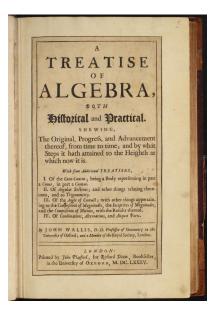
John Wallis, *A treatise of algebra* (1685): complex numbers based on insights derived from

Euclidean geometry

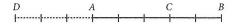
trigonometry

properties of conics

(See: *Mathematics emerging*, §15.1.1.)



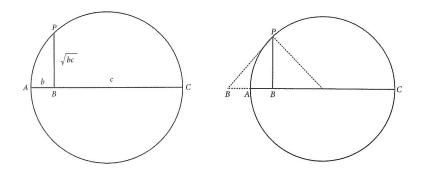
## Wallis: justification of imaginary numbers



- A man starts at A and walks 5 yds to B, then retreats 2 yds to C: overall, he has covered 3 yds. If he instead retreats 8 yds to D, then we may say that he has covered -3 yds.
- Somewhere on the seashore, we gain 26 units of land from the sea, but lose 10 units. Thus, we have gained 16 units overall; if this is a perfect square, then it has side 4 units of length.
- If instead we lose 26 units of land, but gain 10, then we have lost 16 units overall, or gained -16. The area in question (assumed to be a square) might therefore be viewed as having side √-16.

(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 184–185)

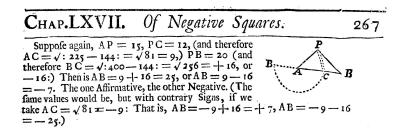
Wallis: imaginary numbers as geometric means



(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 185–186)

## "A new Impossibility in Algebra"

John Wallis, *A treatise of algebra*, p. 267 'Of negative squares': ... requires a new Impossibility in Algebra



Which gives indeed (as before) a double value of AB,  $\sqrt{175}$ ,  $-\sqrt{-81}$ , and  $\sqrt{175}$ ,  $-\sqrt{-81}$ : But fuch as requires a new Impossibility in Algebra, (which in Lateral Equations doth not happen;) not that of a Negative Root, or a Quantity lefs than nothing; (as before,) but the Root of a Negative Square. Which in ftrictness of speech, cannot be: since that no Real Root (Affirmative or Negative,) being Multiplied into itself, will make a Negative Square.

# Complex numbers in the 18th century (1)



Nature remained unclear:

"that amphibian between being and not-being, which we call the imaginary root of negative unity" (Leibniz, 1702)

But complex numbers were increasingly being used ...

# Complex numbers in the 18th century (2)

296 MEMOIRES DE L'ACADEMIE ROYALE boles, dépend en partie de la quadrature du cercle, & en partie de la quadrature de l'hyperbole ou de la defeription de la Logarithmique.

Maniéres abrégées de transformer les différentielles compofées en fimples, & réciproquement; Et même les fimples imaginaires en réelles compofées.

**PROBL.** I. Transformer la différentielle  $\frac{ddz}{bb-zz}$  en une différentielle Logarithmique  $\frac{ddz}{bb}$ , & réciproquement. Faites  $z = \frac{t-1}{t+1}xb_3$  & vous aurez  $\frac{ddz}{bb-zz} = \frac{ddz}{bb}$ . Réciproquement prenez  $t = \frac{t+2}{-z+b}$ , & vous aurez  $\frac{ddz}{bb-zz} = \frac{ddz}{zbr} = \frac{ddz}{bb-zz}$ 

Corol. On transformera de même la différentielle  $\frac{ddz}{b+z}$  en  $\frac{-zdz}{t_b/r-1}$  différentielle de Logarithme imaginaire; & réciproquement.

**PROBL. II.** Transformer la différentielle  $\frac{adz}{bd+zz}$  en différentielle de fecteur ou d'arc circulaire  $\frac{-adz}{a\sqrt{1-bdz}}$ ; & réciproquement.

Faites  $z = \sqrt{\frac{1}{1-bb}}$ , & vous aurez  $\frac{adz}{bb+xz} = \frac{-adt}{y\sqrt{-bbt}}$ Réciproquement prenez  $t = \frac{1}{zz+bb}$ , & vous aurez  $\frac{-adz}{y\sqrt{-bbt}} = \frac{adz}{b+xz}$ .

 $\frac{adz}{B \otimes bL} III. Transformer la différentielle <math>\frac{adz}{bb-zc}$  en différentielle de fecteur hyperbolique  $\frac{adz}{zv't+bbu}$ ; & réciproquement.

Faites  $z = \sqrt{\frac{1}{t} + bb}$ , & enfuite  $t = \frac{1}{bb - zz}$ ; & yous aurez ce qu'on demande. PROBL Johann Bernoulli, 'Solution d'un problème concernant le calcul intégrale, ...', *Mémoires de l'Académie royale des sciences*, 1702:

by making the substitution  $z = \sqrt{\frac{1}{t} - bb}$ , transform the differential  $\frac{adz}{bb+zz}$  into  $\frac{-adt}{2bt\sqrt{-1}}$ 

No worries about the validity of switching between real and complex integrals

(See *Mathematics emerging*, §15.2.1)

## Complex numbers in the 18th century (3)

[ 192 ]

How ÆQUATIONS are to be folu'd.

FTER therefore in the Solution of a Oueflion you are come to an Acquation, and that Acquation is duly reduc'd and order'd ; when the Quantitics which are fuppos'd given, are really given in Numbers, those Numbers are to Se fubfituted in their room in the Equation, and you'll have a Numeral Aquation, whole Root being extracted will fatisfy the Queflion. As if in the Division of an Angle into five equal Parts, by putting r for the Radius of the Circle, a for the Chord of the Complement of the propos'd Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Equation, x'-srrx'+sr\*x-r'q=0. Where in any particular Cafe the Radius r is given in Numbers, and the Line q fubtending the Complement of the given Angle; as if Radius were 10; and the Chord 2; I fubfitute those Numbers in the Equation for r and q, and there comes out the Numeral Aquation x' - 500x' + 50000 - 30000 = o, whereof the Root being extracted will be x, or the Line fubtending the Complement of the fifth Part of that given Angle.

But the Root is a Nomber which being fubfitured in the Equation to the Letter or Species fignifying of the Nature the Root, will make all the lettern vanifufor he Nature of Thus Unity is the Root of the Equation  $x^{-1}$ as Equation:  $-e^{-1} - igx + 4gx - 30 = 0$ , becaufe being write for xit produces 1 - 1 - 1 + 4g

 $-2_0$ , that is, notions. And thus, if for x you write the Number 3, or the Negative Number.  $-x_0$ , and in both Cafes there will be produced nothing, the Affinantive and Negative Terms in thefe four Cafes defronging one another 3; then fince any of the Numbers written in the Equation folish the Convolution of x, by making all the Terms of the Equation together equal to nothing, any of them will be the Root of the Equation.

And that you may not wonder that the fame Equation may have feveral Roots, you mult know that there may be more Solutions [than one] of the fame Problem. As it there was fought the Interfection of two given Circle; there are two Interfections, and confequently the Quefilon admits two Anfwers; and then the Equation determining Isaac Newton, *Universal Arithmetick*, 1728:

p. 195: "it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they are possible" — complex numbers as an indicator of real-world solvability of problems

## Complex numbers in the 18th century (4)

Leonhard Euler also used them freely: e.g., in *Introductio in analysin infinitorum*, 1748, §138:

$$e^{+\nu\sqrt{-1}} = \cos .\nu + \sqrt{-1}.\sin .\nu$$

$$e^{-v\sqrt{-1}} = \cos v - \sqrt{-1} \cdot \sin v$$

(See Mathematics emerging, §9.2.3)

104 DE QUANTATIBUS TRANSCENDENT.
$\frac{L_{1B.L}}{(1+\frac{v\sqrt{-1}}{i})^{i}+(1-\frac{v\sqrt{-1}}{i})^{i}}; \text{ atque for } v =$
$-(1+\frac{1}{i})+(1-\frac{1}{i})$ ; atouc (in $y =$
2
$\frac{\left(1+\frac{v\sqrt{-1}}{i}\right)^{i}-\left(1-\frac{v\sqrt{-1}}{i}\right)^{i}}{v\sqrt{-1}}$ . In Capite autem
. In Capite autem
præcedente vidimus effe $(1 + \frac{3}{i})^i = e^2$ , denotante e balin
Logarithmorum hyperbolicorum : fcripto ergo pro z partim
Logarithmorum hyperbolicorum : for the ergo pro a partim $+ v \sqrt{-1}$ partim $- v \sqrt{-1}$ crit $col v =$ $e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}} & e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}} \\ 2 \sqrt{-1} & k fin v = e^{-v\sqrt{-1}} \\ 2 \sqrt{-1} & v = e^{-v\sqrt{-1}} \\ 2$
$\frac{1}{2}$ $\delta c fin. v = \frac{1}{2\sqrt{-1}}$
Ex quibus intelligitur quomodo quantitates exponentiales ima- ginariæ ad Sinus & Colinus Arcuum realium reducantur. Erit
vero $e^{+\nu\sqrt{-1}} = co(.\nu + \sqrt{-1}.fm.\nu \& e^{-\nu\sqrt{-1}} =$
cof. v - V - I. fin. v.
139. Sit jam in iildem formulis §. 130. # numerus infinite
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$cof. n z = cof. \frac{z}{i} = 1$ & fin. $nz = fin. \frac{z}{i} = \frac{z}{i}$ ; Arcus
enim evanescentis $\frac{z}{i}$ Sinus eft ipfi æqualis, Cofinus vero
= 1. His politis habebitur
$I = \frac{(cof. z + \sqrt{-1}. for. z)^{\frac{1}{1}} + (cof. z - \sqrt{-1}. for. z)^{\frac{1}{1}}}{2} & $
$I = \frac{1}{2} $
÷
$\frac{z}{1} = \frac{(cof.z+\sqrt{-1.fm.z})^{\overline{1}} - (cof.z-\sqrt{-1.fm.z})^{\overline{1}}}{2\sqrt{-1.fm.z}}$ Su-
mendis autem Logarithmis hyperbolicis fupra (125) oftendi-
mus effe $l(1+x) = i(1+x)^{\frac{1}{4}} - i$ , feu $y^{\frac{1}{4}} = 1 + \frac{1}{i}l_j$ ,
mus ene $r(1+x) = r(1+x)^2 = r$ , leu $y^2 = 1 + \frac{1}{r}y$ , polito
ponto

## The Fundamental Theorem of Algebra

Every polynomial equation of degree n has exactly n roots.

- Early 17th century: known that an equation of degree n may have n roots
- During 17th century: complex numbers gradually admitted as roots
- 15 Sept 1759: Euler asserted theorem in a letter to Nicholas Bernoulli, but didn't prove it
- Mid/late 18th century: attempted proofs by Euler, d'Alembert, Lagrange, and others
- 1799: proof by Gauss in his doctoral dissertation, followed by several others
- ▶ 1806: new proof by Argand
- ▶ 1821: Argand's proof appears in Cauchy's *Cours d'analyse*

## New ways of viewing complex numbers



Darwarende Borfog auggare bet Sporgsmadt, fporban Directionen analgetiff ber betegnes, eller hvorban rette Linite burbe ubtraftes, naar af ein eurfte Ligning miltm een ubefindte og ander ginne finite fullte fanne finite et Ubtref, ber forfelliebe baab ben ubefendtures Bengbe og bens Direction.

Bor nagenlebes at finne before bette Sporgemal, legger ig til Gumb webt be Seminger, bet finne im gungettigt. Den freifer es at bes Direce tienned Breanbring, ber ved elgebraitfe Dereationer fan frembringer, agfan ber vod bree Stegn at foreflitte. Den ondens at Direction en togen Blenn fland for Wilgebra, uben for lassibt ben ved algebraitfe Dereationer fan for andere. Mine ab en ved blige i en forsandere i to ken indige feitte ben facwanlige Beetlaring), uben til ben mohatte, eller fan sporte ben friender wander, Beetlaring), uben til ben mohatte, eller fan sporte ben friender Waabbe, og i Genigst til be errige Direktmet støre upsjeldigt. Dette er opjaa Caspar Wessel, 'Om Directionens analytiske Betegning ...' ['On the analytic representation of direction ...'], Nye Samling af det Kongelige Danske Videnskabers Selskabs Skrifter, 1799

Published in Danish not well known

French translation published in 1897

## New ways of viewing complex numbers

### ESSAI

SUR UNE MANIÈRE DE REPRÉSENTER

## LES QUANTITÉS IMAGINAIRES

DASS

LES CONSTRUCTIONS GÉOMÉTRIQUES,

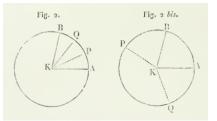
PAR R. ARGAND.

\* ÉDITION Pafcfaöt » <sup>T</sup>ONE PAFACT Par M. J. HOÜEL Fr some sive arressica Gontenant des Extrails des *Annales de Gergones*, relatifs à la question des inscinates.

#### PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE BUBERE DE LOGATEDES, DE L'ÉCOLE POLYTECHNIQUE, SUCCESSER DE MALLET-ACHERIER, Qui de Arguits, 3.

1874 (Tous droits réservés.) Robert Argand, *Essay on a* method of representing imaginary quantities ..., 1806



## New ways of viewing complex numbers

Theory of Conjugate Functions, or Algebraic Couples ; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time.

By WILLIAM ROWAN HAMILTON,

M.R.I. A., F.R.A. S., Hon. M. R. S. Ed. and Dub., Fellow of the American Academy of Arts and Sciences, and of the Royal Northern Antiquarian Society at Copenhagen, Andrews' Professor of Astronomy in the University of Dublia, and Royal Astronome of Ireland.

Read November 4th, 1833, and June 1st, 1835.

General Introductory Remarks.

THE Study of Algebra may be pursued in three very different schools, the Practical, the Philelogical, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation ; according as ease of operation, or symmetry of expression, or clearness of thought, (the agers, the fari, or the supere, / is eminently prized and sought for. The Practical person seeks a Rule which he may apply, the Philological person seeks a Formula which he may write, the Theoretical person seeks a Theorem on which he may meditate. The felt imperfections of Algebra are of three answering kinds. The Practical Algebraist complains of imperfection when he finds his Instrument limited in power ; when a rule, which he could happily apply to many cases, can be hardly or not at all applied by him to some new case; when it fails to enable him to do or to discover something else, in some other Art, or in some other Science, to which Algebra with him was but subordinate, and for the sake of which and not for its own sake, he studied Algebra. The Philological Algebraist complains of imperfection, when his Language presents him with an Anomaly; when he finds an Exception disturb the simplicity of his Notation, or the symmetrical structure of his Syntax : when a Formula must be written with precaution, and a Symbolism is not universal. The Theoretical Algebraist complains of imperfection, when the clearness of his Contemplation is obscured; when the Reasonings of his Science seem anywhere to oppose each other, or become in any part too complex or too little valid for his belief to rest firmly upon them : or when, though trial may have taught him that a rule is useful, or that a formula gives true results, he cannot prove that rule, nor understand that formula : when he cannot rise to intuition from induction, or cannot look beyond the signs to the things signified.

Transactions of the Royal Irish Academy, 1837

Complex numbers as ordered pairs subject to specified rules:

$$(a, b) \pm (c, d) = (a \pm c, b \pm d)$$
$$(a, b)(c, d) = (ac - bd, ad + bc)$$
$$\frac{(a, b)}{(c, d)} = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right)$$

Led to the search for triples, and thence to quaternions

## Complex analysis

The origins of complex analysis may be seen in early achievements by Johann Bernoulli, Euler, and others, using complex transformations to evaluate real integrals. But is substitution of complex variables for real variables permissible?

- Euler (posthumous, 1794): yes
- Laplace (1785, 1812): yes
- ▶ Poisson (1812): doubtful
- Cauchy (1814): inspired by Laplace, set to work on the problem

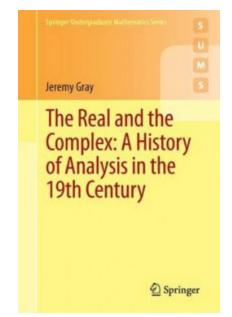
Sources for the origins of complex analysis

Secondary:

- ► Katz: §17.3 (3rd ed.); §22.3 (brief ed.)
- Frank Smithies: Cauchy and the creation of complex function theory, Cambridge University Press, 1997

Primary: as quoted by Smithies; some extracts reproduced in *Mathematics emerging*,  $\S15.2$ .

## Real and complex analysis united



Cauchy as 'creator' of complex analysis

Some of Cauchy's contributions to complex analysis:

- ▶ integration along paths and contours (1814) [1827]
- calculus of residues (1826)
- ▶ integral formulae (1831)
- inferences about Taylor series expansions

 applications to evaluation of difficult definite integrals of real functions Cauchy's changing views of complex numbers and variables

At different times, Cauchy regarded complex numbers in different ways:

- as formal (numerical) expressions  $a + b\sqrt{-1}$ ;
- geometrically;
- ▶ by reducing  $i = \sqrt{-1}$  to a "real but indeterminate quantity"

This done, there is no need to torture the mind to discover what the symbolic sign  $\sqrt{-1}$  could represent . . .

(in modern terms, Cauchy reduced complex arithmetic to calculations modulo  $i^2 + 1$  in  $\mathbb{R}[i]$ )

Moreover, Cauchy's view of complex variables gradually shifted

- from quantities with two parts  $x + y\sqrt{-1}$
- to single quantities z.

## Cauchy's first 'Mémoire' (1814/1827)



#### INTRODUCTION.

La solution d'un grand nombre de problèmes se réduit, en dernière analyse, à l'évaluation des intégrales définies; aussi les géomètres se sont-ils beaucoup occupés de leur détermination. On trouve, à cet égard, une foule de théorèmes curieux et utiles dans les Mémoires et le Calcul intégral d'Euler, dans plusieurs Mémoires de M. Laplace, dans ses Recherches sur les approximations de certaines formules, et dans les Exercices de Calcul intégral de M. Legendre. Mais, parmi les diverses intégrales obtenues par les deux premiers géomètres que je viens de citer, plusieurs ont été découvertes pour la première fois à l'aide d'une espèce d'induction fondée sur le passage du réel à l'imaginaire. Les passages de cette nature conduisent souvent d'une manière très prompte à des résultats dignes de remarque. Toutefois cette portion de la théorie est, ainsi que l'a observé M. Laplace, sujette à plusieurs difficultés. Aussi, après avoir montré, dans le calcul des fonctions génératrices, les ressources que l'Analyse peut retirer de semblables considérations, l'auteur ajoute : « On peut donc considérer ces passages comme des movens de découvertes semblables à l'induction dont les

(1) Ménoires présentes par divers savants à l'Académie royale des Sciences de l'Intitut de France et imprinées par son ordre. Sciences mathématiques et physiques. Tomo I. Imprimé, par autorisation da Rei, à l'Imprimeire royales (2007).

GEorres de C. - S. I. t. I.

Cited Laplace's concerns about the solution of integrals by "the passage from the real to the imaginary"

First part: evaluation of improper integrals, such as

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} \, dx = \frac{\pi}{e}$$

Noted Cauchy–Riemann equations in passing (as had d'Alembert and Euler) as general useful property of analytic functions, rather than fundamental feature of the theory

## Complex numbers in the Cours d'analyse (1821)

176 COURS D'ANALYSE. toute expression symbolique de la forme

a + 61/-1,

 $\alpha$ , C désignant deux quantités réelles ; et l'on dit que deux expressions imaginaires

a+61/-1, ++ 1/-1

sont égales entre elles, lorsqu'il y a égalité de part et d'ante, 1: « entre les parties réelles a et  $\gamma_{i-1}$ , 2.\* entre les coefficiens de  $\sqrt{-1}$ , savoir, C et N. L'égalité de deux expressions imaginaires s'indique, e comme celle de deux quantités réelles, par le signe  $\equiv$ ; et il en résulte ce qu'on appelle une équation imaginaire. Cela posé, toute équation imaginaire n'est que la représentation symbolique de deux équations entre quantités réelles. Par exemple, l'équation symbolique

 $\alpha + 6\sqrt{-1} = \gamma + \delta\sqrt{-1}$ 

équivant seule aux deux équations réelles

 $a = \gamma, \ b = \delta.$ 

Lorsque, dans l'expression imaginaire

a + 6 1/-1,

le coefficient c de  $\sqrt{-i}$  s'évanouit, le terme c  $\sqrt{-i}$ est censé réduit à zéro, et l'expression elle-même à la quantité réelle a. En vertu de cette convention , les expressions imaginaires comprennent, comme cas particuliers, les quantités réelles.

Les expressions imaginaires peuvent être sou-

Defined as "symbolic expressions"  $a + b\sqrt{-1}$ 

55-page development of formal definitions and properties

Consideration of multi-functions — which are the most natural branches to take?

Sought to extend ideas for real functions to the complex case, particularly those relating to power series and convergence

## Cauchy's second 'Mémoire' (1825)

'Mémoire sur les intégrales définies, prises entre des limites imaginaires'

Direct adaptation of definition of real integral to the complex case:

$$\int_{x_0+y_0\sqrt{-1}}^{X+Y\sqrt{-1}}f(z)dz$$

is the limit (or one of the limits) of a sum of products of the form

$$\sum (x_{i-1} + y_{i-1}\sqrt{-1})f(x_{i-1} + y_{i-1}\sqrt{-1}).$$

NB. No explicit definition of a function of a complex variable; tacit assumption of differentiability, hence that the Cauchy–Riemann equations hold.

## Contour integration

In any domain where the function does not become infinite, the value of a complex integral is independent of the path along which it is taken.

Cauchy: consider two different paths within the rectangle  $(x_0, y_0)$ , (X, Y) such that the function  $f(x + y\sqrt{-1})$  does not become infinite for values of x, y lying within the domain enclosed by the paths. Then the value of the integral  $\int_{x_0+y_0}^{X+Y\sqrt{-1}} f(z)dz$  is independent of the path taken.

Really a theorem about real functions in the plane?

(Gauss had discovered this in 1811, alongside a similar definition of a complex integral, but did not publish.)

## Contour integration

For the case where  $f(x + y\sqrt{-1})$  becomes infinite at the point x = a, y = b, Cauchy considered the limit

$$\mathsf{f} := \lim_{\substack{x \to a \\ y \to b}} \left( x - a + (y - b)\sqrt{-1} \right) f\left( x + y\sqrt{-1} \right),$$

and determined that the difference between the integrals of f along different paths that are infinitely close to each other as well as to (a, b) is  $2\pi f \sqrt{-1}$ .

With a natural extension of this result for multiple and/or higher-order singularities, this became an ancestor of Cauchy's residue theorem — developed as part of Cauchy's calculus of residues in a paper of 1826 ('Sur un nouveau genre de calcul'). Taylor's Theorem for complex analytic functions

In *Cours d'analyse* (1821), Cauchy had considered the notion of radius of convergence for both real and imaginary power series.

1831: a complex function has a convergent power series if it is "finite and continuous"

Continued to refine the conditions for the theorem over many years.

Cauchy's language is not always satisfactory to modern eyes, but was considerably more rigorous than that of most of his contemporaries.

1841: extension to negative powers — Laurent's Theorem.

## Cauchy's complex analysis

Cauchy's ideas concerning complex functions developed over many years. In the early stages

- did he appreciate the fundamental nature of the concepts and results that he was using and deriving?
- did he recognise the subtleties of working with complex numbers rather than simply with pairs of real numbers?

Have historians of mathematics read too much into the earlier work on the basis of what came later?

Point to note: Cauchy may be credited with many of the fundamental ideas of complex analysis, but this does not mean that they appeared fully-formed.

## Riemann on complex analysis

### GRUNDLAGEN

FÜR EINE

### ALLGEMEINE THEORIE DER FUNCTIONEN

EINER

#### VERÄNDERLICHEN COMPLEXEN GRÖSSE.

(Georg Friedrich) Bernhard B. RIEMANN. •

WEITER, UNVERÄNDERTER ABDRUCE.

<sup>C</sup>GÖTTINGEN, verlag von adalbert rente. 1867. Doctoral dissertation: Foundations for a General Theory of Functions of a Variable Complex Quantity (1851)

Started from the idea that a complex variable should be treated as a single quantity z

"The complex variable w is called a function of another complex variable z when its variation is such that the value of the derivative  $\frac{dw}{dz}$  is independent of the value of dz"

That is:  $\lim_{\delta \to 0} \frac{f(z+\delta)-f(z)}{\delta}$  exists

### Riemann on complex analysis

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so erhellt, dass er und zwar nur dann für je zwei Werthe von dx und dy denselben Werth haben wird, wenn

$$\frac{du}{dx} = \frac{dv}{dy}$$
 und  $\frac{dv}{dx} = -\frac{du}{dy}$ 

ist. Diese Bedingungen sind also hinreichend und nothwendig, damit w = u + vi eine Function von z = x + yi sei. Für die einzelnen Glieder dieser Function fliessen aus ihnen die folgenden :

$$\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = o, \quad \frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} = o$$

wicht für die Unternuchung der Eigenschaften, die Einem Gibele einer solchen Function einsaln betrachtet zukommen, die Grundlage bilden. Wir werden den Beweis für die wichtligsten dieser Eigenschaften einer eingebenderen Betrachtung der volkistachigen Function vorzufgelein lasten, zwer aber noch einige Punkte, welche allgemeinzem Gebieten angebören, erörtern und ferdagen, um uns den Boden für genu Unterenchungen zu ebenen.

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Per dis folgenden Betrachtangen beschrächen wir die Verstachtfahlteit der Gressen z. y. zur im onflichen Geloht, indem vir als der die Arnaktes 0 als die nach die Eissen A solltet, nondren dies ther disselbe ausgebreitete Filzekar Teitrachten. Wir willam diese Kinkleitung, ein die ausnatzleitung ein wirk, von aufnichunge lingesden Filzekan zurohen, um die Möglichkeit offen zu hassen, dass der Ort das Paaktes O there dessallen Tahil der Ebens ein dem Artidie derrechtes; stehn zufeich für ein schauften alle Stehn Stehn diese Filzekanthälle sinder Filzekan zu einer Zufeitung einer Pallerung, dass dass Umfahlung der Filzeka, oder eines Spallung in und einsten Filsekan und ein sicht verkommt.

Die Anzahl der in jedem Theile der Ebene auf einzahler liegenden Flichentheile ist alsekann vollkommen bestimmt, wenn die Begrennung der Lage und dem Sinne nach (d. h. ihre innere und aussere Seite) gegeben zist jir Verlauf kann sich jedoch noch verthöleten gestalten.

In due That, ishen wir durch den von der Fisch bedekten Theil der Ebens eine beliebig lähel 1, os aufset sich die Annall der ther einauen liegender Fischenbelle aur beitungengesegnetten der Bleur m. – 1, um eingergesegnetten der Bleur m. – 1, um eingergesegnetten der Bleur m. – 1, um eingergesegnetten der Bleur m. – 1, die auf als als ob berall beitunk. Lasg ed u. Uber dieser Liefe der Bleur m. – 1, die eine Bleur m. – 1, die Bleur m. – 1, die eine Bleur m. – 1, die Bleur m. – 1, die eine Bleur m. – 1, die Bleur Bleur Bleur m. – 1, die Bleur Bleur Mehren Bleur Mehren Bleur Mehren Bleur Bleur Mehren Bleur Bleur m. – 1, die Bleur B

Cauchy–Riemann equations now taken as fundamental to the theory

Other key concepts appear explicitly:

- harmonic functions;
- conformality (a complex function preserves angles wherever its derivative does not vanish);

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Early impact limited by abstraction and restricted publication

## The word 'analytic'

The words analysis, analytic have had many meanings:

- Classical: a method of investigating a problem, the opposite of synthesis
  - c. 1600: algebra became known as the 'analytic art' or just 'analysis', using finite equations
    - 1669: Newton introduced 'analysis with infinite equations', that is, infinite series
    - 1748: Euler wrote on the analysis of infinitely large and infinitely small quantities
- 1790–1840: in sections of journals, the Académie des Sciences, etc., Analyse could mean 'pure mathematics' though with a bias to algebra, calculus, etc.; compare Géométrie also meaning 'pure mathematics', but with (perhaps) spatial bias
  - 1821: Cauchy's cours d'analyse shows similarities with our analysis courses today

## What *is* an analytic function?

Lagrange, 1797: function is analytic if it has a power-series expansion

Cauchy's point of departure, 1814–1831: treated complex functions that are continuous and satisfy the Cauchy–Riemann equations (always true for analytic functions in the sense of Lagrange), but used no special terminology

Riemann, 1851: switched focus to complex functions for which  $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$  exists in the region of interest

Weierstrass, 1860s: applied Lagrange's term analytic to Riemann's conception of function

Oxford, 2018: we follow Riemann and Weierstrass, by using the words holomorphic, meromorphic, etc. as variants of analytic, with slightly different meanings