Model solutions and marking scheme for B4.4. **March 2021**

Question 1: (a) Since a, $b \in \mathrm{L}^1(\mathbb{R})$ the rule, $\langle a, \phi \rangle := \int_{\mathbb{R}} a(x)\phi(x)\,\mathrm{d}x$ and $\langle b, \phi \rangle := \int_{\mathbb{R}} b(x)\phi(x)\,\mathrm{d}x$ for $\phi \in \mathscr{S}(\mathbb{R})$ yield well-defined linear functionals on $\mathscr{S}(\mathbb{R})$. Furthermore, $\left|\langle a, \phi \rangle\right| \leq \|a\|_1 \mathrm{S}_{0,0}(\phi)$ and similarly for b, so both are $\mathscr{S}$ continuous, and thus tempered distributions. For c we note that $c(x)/(1+|x|) \in \mathrm{L}^1(\mathbb{R})$, so c is a tempered L^1 function and is thus a tempered distribution by the definition $\langle c, \phi \rangle = \int_{\mathbb{R}} c(x)\phi(x)\,\mathrm{d}x$ in view of the bound $\left|\langle c, \phi \rangle\right| \leq 2\|c/(1+|\cdot|)\|_1 \overline{S}_{1,0}(\phi)$. **[2 marks]**

The Fourier transform of a is easily calculated

$$\widehat{a}(\xi) = \int_0^\infty \mathrm{e}^{-(1+\mathrm{i}\xi)x}\,\mathrm{d}x = \frac{1}{1+\mathrm{i}\xi} = \frac{-\mathrm{i}}{\xi - \mathrm{i}}.$$

[1 mark]

Note that

$$\frac{\mathrm{i}}{\xi + \mathrm{i}} + \frac{-\mathrm{i}}{\xi - \mathrm{i}} = \frac{2}{1+\xi^2},$$

hence $b = \frac{1}{2}\big(\widetilde{\widehat{a}} + \widehat{a}\big)$ and so by the Fourier inversion formula in $\mathscr{S}'$,

$$\widehat{b}(\xi) = \frac{1}{2}\big(2\pi\widetilde{\widetilde{a}} + 2\pi\widetilde{a}\big) = \pi\mathrm{e}^{-|\xi|}.$$

[2 marks]

Since $c(x) = xb(x)$ we get by the differentiation rule,

$$\widehat{c}(\xi) = \mathrm{i}\widehat{b}'(\xi) = -\mathrm{i}\pi\mathrm{e}^{-|\xi|}\mathrm{sgn}(\xi).$$

[2 marks]

[Standard examples + seen related examples before.]

(b) Using the dilation rules and (a), we have in $\mathscr{S}'(\mathbb{R})$,

$$\widehat{b_\varepsilon}(\xi) = \widehat{b}(\varepsilon\xi) = \pi\mathrm{e}^{-\varepsilon|\xi|} \to \pi\mathbf{1}_{\mathbb{R}} \text{ as } \varepsilon \searrow 0.$$

[1 mark]

Hence by the Fourier inversion formula in $\mathscr{S}'$ and $\mathscr{S}'$ continuity of $\mathcal{F}^{-1}$,

$$b_\varepsilon = \mathcal{F}^{-1}_{\xi\to x}\big(\pi\mathrm{e}^{-\varepsilon|\xi|}\big) \to \mathcal{F}^{-1}\big(\pi\mathbf{1}_{\mathbb{R}}\big) = \pi\delta_0 \text{ in } \mathscr{S}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0.$$

[1 mark]

For c_ε we get similarly,

$$\widehat{c_\varepsilon}(\xi) = \widehat{c}(\varepsilon\xi) \to -\mathrm{i}\pi\mathrm{sgn}(\xi) \text{ in } \mathscr{S}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0,$$

hence

$$c_\varepsilon(x) \to \mathcal{F}^{-1}_{\xi\to x}\big(-\mathrm{i}\pi\mathrm{sgn}(\xi)\big) \text{ in } \mathscr{S}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0.$$

[1 mark]

From an example in course lecture notes (or by calculation), $\widehat{\mathrm{pv}\big(\frac{1}{x}\big)}(\xi) = -\mathrm{i}\pi\mathrm{sgn}(\xi)$ so we find the limit of c_ε is $\mathrm{pv}\big(\frac{1}{x}\big)$. **[1 mark]**
(It is ok to normalize by π and use results about approximate units from the course to find limit for b_ε. One can also quite easily find limit of c_ε without use of Fourier transform.)
For $z = x + \mathrm{i}y$ in the upper half-plane we have

$$F(z) = \frac{y}{x^2+y^2} + \mathrm{i}\frac{x}{x^2+y^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z}$$

so clearly holomorphic. **[1 mark]**
For real-valued $\varphi \in \mathscr{S}(\mathbb{R})$ we define

$$\Phi(z) = \tfrac{1}{\pi}\big((b_y * \varphi)(x) + \mathrm{i}(c_y * \varphi)(x)\big)$$

[2 marks]

and since we can rewrite this as

$$\Phi(z) = \frac{1}{\pi}\big\langle F(z-\cdot), \varphi\big\rangle$$

[1 mark]

it follows from a theorem about differentiation behind the distribution sign that Φ is C^1 and satisfies the Cauchy-Riemann equation, and hence that it is holomorphic on the upper half-plane. **[1 mark]**
Since φ is real-valued, $\mathrm{Re}(\Phi(z)) = (b_y * \varphi)(x)/\pi$ and so by the first part of (b) we get that it converges to $\varphi(x)$ pointwise in $x \in \mathbb{R}$ as $y \searrow 0$. **[1 mark]**
For the imaginary part,

$$\mathrm{Im}\big(\Phi(z)\big) = \frac{1}{\pi}\big(c_y * \varphi\big)(x) \to \frac{1}{\pi}\bigg(\mathrm{pv}\big(\frac{1}{t}\big) * \varphi\bigg)(x) = \mathcal{H}(\varphi)(x)$$

pointwise in $x \in \mathbb{R}$ as $y \searrow 0$, where $\mathcal{H}$ is the Hilbert transform. **[2 marks]**
[Seen variants before]
(c) Note that the function $f - c$ is continuous and that

$$f(x) - c(x) = \mathrm{O}\big(\frac{1}{x^2}\big) \text{ as } |x| \to \infty.$$

[2 marks]

It follows that $f - c \in \mathrm{L}^1(\mathbb{R})$ and so by the Riemann-Lebesgue lemma, $\widehat{(f-c)} \in \mathrm{C}_0(\mathbb{R})$. **[1 mark]**

Thus

$$\widehat{f}(\xi) = \widehat{(f-c)}(\xi) - \mathrm{i}\pi \mathrm{e}^{-|\xi|}\mathrm{sgn}(\xi).$$

[1 mark]

Consequently $\widehat{f}$ is continuous at each $\xi \neq 0$ and at 0 it has one-sided limits $\widehat{f}(0^-) = \mathrm{i}\pi$, $\widehat{f}(0^+) = -\mathrm{i}\pi$. Hence it has a jump discontinuity at 0 with jump $-2\pi\mathrm{i}$. **[2 marks]**

[**New example**]

Question 2: (a) (i) The symbol for $p(\partial) = -\Delta - \mathrm{i}\partial_1\partial_2 + 1$ is the polynomial $p(\mathrm{i}\xi) = \xi_1^2 + \xi_2^2 + \mathrm{i}\xi_1\xi_2 + 1$ and the principal symbol is $\xi_1^2 + \xi_2^2 + \mathrm{i}\xi_1\xi_2$. It is clear that the principal symbol only vanishes at $\xi = 0$ in $\mathbb{R}^2$, so the differential operator is elliptic.

[2 marks]

A fundamental solution for $p(\partial)$ is any distribution $E \in \mathscr{D}'(\mathbb{R}^2)$ satisfying $p(\partial)E = \delta_0$ in $\mathscr{D}'(\mathbb{R}^2)$. We consider this equation in $\mathscr{S}'(\mathbb{R}^2)$: if $E \in \mathscr{S}'(\mathbb{R}^2)$ and $p(\partial)E = \delta_0$ in $\mathscr{S}'(\mathbb{R}^2)$, then by Fourier transformation and the differentiation rule, $p(\mathrm{i}\xi)\widehat{E} = 1$ in $\mathscr{S}'(\mathbb{R}^2)$. Because the symbol satisfies

$$\left|p(\mathrm{i}\xi)\right| = \sqrt{(\xi_1^2 + \xi_2^2 + 1)^2 + (\xi_1\xi_2)^2} \geq \xi_1^2 + \xi_2^2 + 1 = |\xi|^2 + 1,$$

it follows that $\frac{1}{p(\mathrm{i}\xi)}$ is a tempered L^1 function and so in particular a tempered distribution. We must therefore have that $\widehat{E} = \frac{1}{p(\mathrm{i}\xi)}$, and hence by the Fourier inversion formula in $\mathscr{S}'$ that

$$E = \mathcal{F}^{-1}\left(\frac{1}{p(\mathrm{i}\xi)}\right) \in \mathscr{S}'(\mathbb{R}^2)$$

is uniquely determined. Conversely, it is easy to check that this is indeed a fundamental solution. **[2 marks]**

(ii) First note,

$$\partial_1\widehat{E} = -\frac{2\xi_1 + \mathrm{i}\xi_2}{p(\mathrm{i}\xi)^2} = -(2\xi_1 + \mathrm{i}\xi_2)\widehat{E}^2,$$

and so

$$\partial_1\widehat{E}^k = k\widehat{E}^{k-1}\partial_1\widehat{E} = -k(2\xi_1 + \mathrm{i}\xi_2)\widehat{E}^{k+1}.$$

Using Leibniz' rule we then find for $m > 1$,

$$\begin{aligned}\partial_1^m \widehat{E}^k &= -k\partial_1^{m-1}\Big((2\xi_1 + \mathrm{i}\xi_2)\widehat{E}^{k+1}\Big)\\ &= -k\sum_{j=0}^{m-1}\binom{m-1}{j}\partial_1^j(2\xi_1+\mathrm{i}\xi_2)\partial_1^{m-1-j}\widehat{E}^{k+1}\\ &= -k\Big((2\xi_1+\mathrm{i}\xi_2)\partial_1^{m-1}\widehat{E}^{k+1} + 2(m-1)\partial_1^{m-2}\widehat{E}^{k+1}\Big)\end{aligned}$$

as required. Next, note that for $k \in \mathbb{N}$,

$$\big|\widehat{E}^k\big| \le \big(1+|\xi|^2\big)^{-k} \text{ and } \big|\partial_1\widehat{E}^k\big| \le 2k\big(1+|\xi|^2\big)^{-k+\frac{1}{2}}.$$

Thus we have $c(k,0) = 1$ and $c(k,1) = 2k$. Assume that for some $s \in \mathbb{N}$ we have the inequality for $m \le s$ and all $k \in \mathbb{N}$. Then we get from the above recurrence relation, the triangle inequality and the induction hypothesis:

$$\begin{aligned}\big|\partial_1^{s+1}\widehat{E}^k\big| &\le k\big(2|\xi_1|+|\xi_2|\big)\big|\partial_1^s\widehat{E}^{k+1}\big| + 2ks\big|\partial_1^{s-1}\widehat{E}^{k+1}\big|\\ &\le \big(2kc(k+1,s)+2ksc(k+1,s-1)\big)\big(1+|\xi|^2\big)^{-k-\frac{s+1}{2}}\end{aligned}$$

This is the required bound with $c(k,s+1) = 2kc(k+1,s)+2ksc(k+1,s-1)$. The assertion now follows by induction. **[5 marks]**
(iii) From (ii) we have for any multi-index α,

$$\big|\xi^\alpha\partial_1^m\widehat{E}\big| \le |\xi^\alpha| c_m\big(1+|\xi|^2\big)^{-1-\frac{m}{2}} \le c_m\big(1+|\xi|^2\big)^{\frac{|\alpha|-m-2}{2}},$$

hence we have integrability over $\mathbb{R}^2$ provided $|\alpha| < m$. **[2 marks]**
By the differentiation rules and the Riemann-Lebesgue lemma we therefore have

$$\partial^\alpha\Big(\big(-\mathrm{i}x_1\big)^m E\Big) = \mathcal{F}^{-1}_{\xi\to x}\Big(\big(\mathrm{i}\xi\big)^\alpha\partial_1^m\widehat{E}\Big) \in \mathrm{C}_0(\mathbb{R}^2)$$

provided $|\alpha| < m$. Consequently, the function $x_1^m E$ is $C^{m-1}(\mathbb{R}^2)$ and so E is C^{m-1} away from the x_2-axis. Since $m \in \mathbb{N}$ was arbitrary, we have shown that E is C^∞ away from the x_2-axis. **[2 marks]**
Now note that $\widehat{E}$ is symmetric in ξ_1 and ξ_2, so that we can do exactly the same calculation with respect to ξ_2 to see that E is C^∞ away from the x_1-axis. We then conclude that E is C^∞ away from the origin, and hence that $\mathrm{sing.supp}(E) \subseteq \{0\}$. **[2 marks]**
[(i) is straightforward. (ii) and (iii) are variants of a calculation for Bessel kernels done in course]

(b)(i) The PDE is by the Fourier inversion formula in $\mathscr{S}'$ equivalent to $p(\mathrm{i}\xi)\widehat{U} = \widehat{F}$, and because $\widehat{E} = 1/p(\mathrm{i}\xi)$ in particular is a moderate C^∞ function we can rewrite this as $\widehat{U} = \widehat{E}\widehat{F}$, and so by the extended convolution rule and Fourier inversion,

$$U = E * F \in \mathscr{S}'(\mathbb{R}^2)$$

is the unique solution in $\mathscr{S}'(\mathbb{R}^2)$. Now

$$\big|(1+|\xi|^2)\widehat{U}\big| = \big|(1+|\xi|^2)\widehat{E}\big||\widehat{F}| \leq |\widehat{F}|,$$

so by the definition of the H^2 norm and Plancherel's theorem,

$$\|u\|_{\mathrm{H}^2} \leq \|\widehat{F}\|_2 = 2\pi\|F\|_2,$$

as required. **[4 marks]**

(ii) Suppose that $u \in \mathscr{D}'(\Omega)$ is a solution. Fix $\omega \Subset \Omega$ and let $\chi = \rho_\varepsilon * \mathbf{1}_{B_\varepsilon(\omega)}$ for $\varepsilon > 0$ so small that $\chi \in \mathscr{D}(\Omega)$. If we define $\chi f = 0$ off Ω, then $\chi f \in \mathrm{L}^2(\mathbb{R}^2)$ and we can use (i) with $F = \chi f$ to assert that $U = E*(\chi f) \in \mathrm{H}^2(\mathbb{R}^2)$ satisfies $p(\partial)U = \chi f$ in $\mathscr{S}'(\mathbb{R}^2)$. Because $\chi = 1$ on ω we have for $\phi \in \mathscr{D}(\omega)$,

$$\big\langle p(\partial)(u-U), \phi\big\rangle = \big\langle f - \chi f, \phi\big\rangle = \big\langle f, (1-\chi)\phi\big\rangle = 0$$

and thus, $p(\partial)(u - U) = 0$ in $\mathscr{D}'(\omega)$. By (a)(iii) and a result from the course the differential operator $p(\partial)$ is hypoelliptic, so $u-U$ is C^∞ on ω. It follows that u is locally H^2 on ω, and since $\omega \Subset \Omega$ was arbitrary the proof is complete. **[6 marks]**

[New example]

Question 3: (a) Let $\chi = \rho * \mathbf{1}_{(-1,2\pi+1]}$ and note that the 2π periodisation of χ, $P\chi$, is a C^∞ function with $P\chi \geq 1$. If $\Psi = \chi/P\chi$, then $\Psi \in \mathscr{D}(\mathbb{R})$ with periodisation $P\Psi = 1$. The 2π periodicity of u implies that $\langle u, \phi\rangle = \langle u, \Psi\phi\rangle$ holds for all $\phi \in \mathscr{D}(\mathbb{R})$, and this formula can be used to extend u to $\mathscr{S}(\mathbb{R})$ as a tempered distribution. (Candidates need not mention this.) The Fourier expansion of u is

$$u = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx}, \quad c_k = \frac{1}{2\pi}\big\langle u, \Psi \mathrm{e}^{-\mathrm{i}k(\cdot)}\big\rangle$$

The convergence is in the sense of $\mathscr{S}'$:

$$\sum_{k=-m}^{n} c_k \mathrm{e}^{\mathrm{i}kx} \to u \ \text{ in } \ \mathscr{S}'(\mathbb{R})$$

as $m,\, n \to \infty$. We have uniqueness in the sense that when $\sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx} = 0$ in $\mathscr{S}'(\mathbb{R})$, then $c_k = 0$ for all $k \in \mathbb{Z}$. **[Book work. 2 marks]**

(i) $(c_k)_{k\in\mathbb{Z}}$ is the sequence of Fourier coefficients for a 2π periodic distribution if and only if there exist constants $c \geq 0$, $N \in \mathbb{N}_0$ such that

$$|c_k| \leq c\big(1+|k|^2\big)^{\frac{N}{2}}$$

holds for all $k \in \mathbb{Z}$. Now assume $u = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx}$ in $\mathscr{S}'(\mathbb{R})$. If $\phi \in \mathscr{S}(\mathbb{R})$ we have, since $\widehat{\phi} \in \mathscr{S}(\mathbb{R})$ too, that

$$\big|c_k\langle \mathrm{e}^{\mathrm{i}kx}, \phi\rangle\big| \leq |c_k\widehat{\phi}(-k)| \leq c\big(1+|k|^2\big)^{\frac{N}{2}}|\widehat{\phi}(-k)| \leq c2^{N+2}\big(1+|k|^2\big)^{-1}\overline{S}_{N+2,0}(\widehat{\phi})$$

for all $k \in \mathbb{Z}$, hence

$$\sum_{k\in\mathbb{Z}}\big|c_k\langle \mathrm{e}^{\mathrm{i}kx}, \phi\rangle\big| \leq c2^{N+2}\sum_{k\in\mathbb{Z}}\big(1+|k|^2\big)^{-1}\overline{S}_{N+2,0}(\widehat{\phi}) < \infty.$$

The series $\sum_{k\in\mathbb{Z}} c_k\langle \mathrm{e}^{\mathrm{i}kx}, \phi\rangle$ is therefore absolutely convergent, and consequently

$$\sum_{k\in\mathbb{Z}} c_{\sigma(k)}\langle \mathrm{e}^{\mathrm{i}\sigma(k)x}, \phi\rangle = \langle u, \phi\rangle$$

as required. **[4 marks]**

(ii) If $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ is 2π periodic, then we have for $k \neq 0$ that $\mathrm{e}^{-\mathrm{i}kx} = -\mathrm{e}^{-\mathrm{i}k(x-\frac{\pi}{k})}$ and hence

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = -\frac{1}{2\pi}\int_0^{2\pi} f(x+\frac{\pi}{k})\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

Therefore

$$c_k = \frac{1}{4\pi}\int_0^{2\pi}\big(f(x) - f(x+\frac{\pi}{k})\big)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

Consequently $|c_k| \leq \frac{1}{4\pi}\int_0^{2\pi}|f(x) - f(x+\frac{\pi}{k})|\,\mathrm{d}x \to 0$ as $|k| \to \infty$. **[4 marks]**

(iii) We have $v = \sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}kx}$ in $\mathscr{S}'(\mathbb{R})$. Because $\sum_{k\in\mathbb{Z}}|c_k| < \infty$ the Weierstrass M-test implies the Fourier series converges uniformly, and hence it follows that the function $f(x) := \sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}kx}$ is continuous. Now for $\phi \in \mathscr{S}(\mathbb{R})$ we have by definition and by uniform convergence that

$$\langle v, \phi\rangle = \left\langle \sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}kx}, \phi\right\rangle = \langle f, \phi\rangle$$

and so $v = f$ as tempered distributions. In particular, $\|v\|_\infty = \max_{x\in[0,2\pi]} |f(x)| = |f(x_0)|$ for $x_0 \in [0, 2\pi]$ say, and so

$$\|v\|_\infty = |f(x_0)| = \left|\sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx_0}\right| \le \sum_{k\in\mathbb{Z}} |c_k|,$$

as required. **[4 marks]**

[(i) new example, but routine. (ii) seen before. (iii) new example, but routine]

(b)(i) Put $p_n(x) = 1 + a_n \cos\big(3^n x + \theta_n\big)$. Then

$$p_n(x) = 1 + \frac{a_n}{2}\mathrm{e}^{\mathrm{i}\theta_n}\mathrm{e}^{\mathrm{i}3^n x} + \frac{a_n}{2}\mathrm{e}^{-\mathrm{i}\theta_n}\mathrm{e}^{-\mathrm{i}3^n x}$$

and so we get by inspection and since the representation of each element in Λ is unique,

$$p(x) = \prod_{n=1}^{N} p_n(x) = \sum_{\lambda\in\Lambda}\left(\prod_{n=1}^{N}\left(\frac{a_n}{2}\right)^{|\varepsilon_n|}\mathrm{e}^{\mathrm{i}\varepsilon_n\theta_n}\right)\mathrm{e}^{\mathrm{i}\lambda x}.$$

We infer from this that the Fourier coefficients $A_k = 0$ when $k \notin \Lambda$.

[3 marks]

For $\lambda = 0 \in \Lambda$ we have $\varepsilon_n = 0$ for all n, and so $A_0 = 1$. Since $\lambda = 3^m \in \Lambda$ when $1 \le m \le N$ we have $\varepsilon_n = \delta_{m,n}$, and so $A_{3^m} = \frac{a_m}{2}\mathrm{e}^{\mathrm{i}\theta_m}$. **[2 marks]**

(ii) Take for each $n \in \{1, \ldots, N\}$,

$$a_n = 1 \text{ and } \theta_n = \mathrm{Arg}(c_n).$$

Define

$$p(x) = \prod_{n=1}^{N}\Big(1 + a_n \cos\big(3^n x + \theta_n\big)\Big).$$

[1 mark]

Then we have $p(x) \ge 0$ and according to (i),

$$p(x) = \sum_{\lambda\in\Lambda} A_\lambda \mathrm{e}^{\mathrm{i}\lambda x},$$

where $A_0 = 1$, $A_{3^n} = \frac{1}{2}\mathrm{e}^{\mathrm{i}\mathrm{Arg}(c_n)}$ for $n \in \{1, \ldots, N\}$. In particular we record that $\frac{1}{2\pi}\int_0^{2\pi} |p(x)|\,\mathrm{d}x = 1$. **[1 mark]**

By virtue of Plancherel's theorem for Fourier series we get

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=1}^{N} c_n \mathrm{e}^{\mathrm{i}3^n x}\right)\overline{p(x)}\,\mathrm{d}x = \sum_{n=1}^{N} c_n \frac{1}{2}\mathrm{e}^{-\mathrm{i}\mathrm{Arg}(c_n)} = \frac{1}{2}\sum_{n=1}^{N} |c_n|,$$

and consequently

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{N}|c_n| &\leq \sup_{x\in(0,2\pi]}\left|\sum_{n=1}^{N} c_n \mathrm{e}^{\mathrm{i}3^n x}\right| \frac{1}{2\pi}\int_0^{2\pi}|p(x)|\,\mathrm{d}x \\ &= \sup_{x\in(0,2\pi]}\left|\sum_{n=1}^{N} c_n \mathrm{e}^{\mathrm{i}3^n x}\right|\end{aligned}$$

as required. **[2 marks]**

If g is a 2π periodic L^∞ function with Fourier expansion

$$g(x) = \sum_{n=1}^{\infty} c_n \mathrm{e}^{\mathrm{i}3^n x} \ \text{ in } \ \mathscr{S}'(\mathbb{R}),$$

then for $N \in \mathbb{N}$ we can define $p(x)$ as above corresponding to the N-th partial sum. Since in particular $g \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R})$ and $3^n \notin \Lambda$ for $n > N$, we infer from Plancherel's theorem for Fourier series that

$$\frac{1}{2\pi}\int_0^{2\pi} g(x)\overline{p(x)}\,\mathrm{d}x = \frac{1}{2}\sum_{n=1}^{N}|c_n|,$$

and therefore that

$$\frac{1}{2}\sum_{n=1}^{N}|c_n| \leq \|g\|_\infty \frac{1}{2\pi}\int_0^{2\pi}|p(x)|\,\mathrm{d}x = \|g\|_\infty.$$

Because $N \in \mathbb{N}$ is arbitrary we are done. **[2 marks]**

[New example]