C5.9

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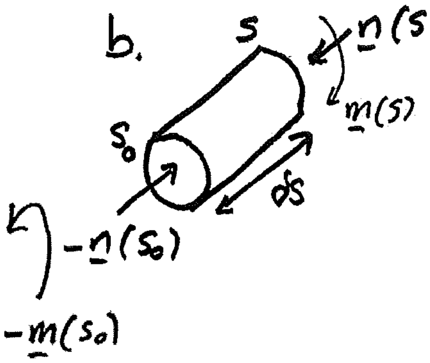
Q1.

a. $$\frac{\partial \underline{d_i}}{\partial s} = \underline{u} \wedge \underline{d_i} = \left[u_1 \underline{d_1} + u_2 \underline{d_2} + \underbrace{u_3 \underline{d_3}}\right] \wedge d_i$$

u_3 rotates $\underline{d_1}, \underline{d_2}$ about $\underline{d_3}$, twisting the filament

$\therefore$ No twist implies $u_3 = 0$ 2

b. 

On this filament section a force balance gives

$$\underline{n}(s_0 + \delta s) - \underline{n}(s_0) + \underline{f}(s_0)\delta s = 0$$ 2

static $\therefore$ no d^2/dt^2 term.

$$\therefore \quad \frac{\partial \underline{n}}{\partial s} + \underline{f} = 0$$ 2

A moment balance gives

$$\underline{0} = \underline{m}(s_0 + \delta s) - \underline{m}(s_0) + (\underline{r} \wedge \underline{n})(s_0 + \delta s) - \underline{r} \wedge \underline{n}(s_0) + \underline{l}(s_0)\delta s + \underline{r} \wedge \underline{f}(s_0)\delta s$$ 3

$$\therefore \quad 0 = \frac{\partial \underline{m}}{\partial s} + \frac{\partial}{\partial s}(\underline{r} \wedge \underline{n}) + \underline{r} \wedge \underline{f} + \underline{l} = \frac{\partial \underline{m}}{\partial s} + \underline{r} \wedge \left(\frac{\partial \underline{n}}{\partial s} + \underline{f}\right) + \frac{\partial \underline{r}}{\partial s} \wedge \underline{n} + \underline{l}$$

$$= \frac{\partial \underline{m}}{\partial s} + \frac{\partial \underline{r}}{\partial s} \wedge \underline{n} + \underline{l},$$ using force balance above. 3

c) i) $$\underline{d_3} = (\cos\theta, \sin\theta, 0)$$

$$\underline{d_1} = \underline{d_2} \wedge \underline{d_3} = \begin{vmatrix} \underline{e_1} & \underline{e_2} & \underline{e_3} \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = (-\sin\theta, \cos\theta, 0)$$

$$\therefore\ 0 = \frac{d\underline{d_2}}{ds} = \underline{u}\wedge \underline{d_2} = (u_1\underline{d_1}+u_2\underline{d_2})\wedge\underline{d_2} = u_1\underline{d_3} \quad \therefore u_1 = 0$$ 1

$$\frac{d\underline{d_3}}{ds} = \theta_s(-\sin\theta, \cos\theta, 0) = (\underbrace{u_1\underline{d_1}}_{0}+u_2\underline{d_2})\wedge\underline{d_3} = u_2\underline{d_1}$$

$$= u_2(-\sin\theta,\cos\theta,0)$$

2

$$\therefore u_2 = \theta_s$$

To find n_2, consider moment balance

$$0 = \frac{\partial \underline{M}}{\partial s} + \frac{\partial \underline{r}}{\partial s}\wedge\underline{n} = E\theta_{ss}\underline{d_2} + \underline{d_3}\wedge(n_1\underline{d_1}+n_2\underline{d_2})$$

$$= E\theta_{ss}\underline{d_2} + n_1\underline{d_2} - n_2\underline{d_1}$$

2

$\therefore$ From $\underline{d_1}$ coefficient, $n_2 = 0$.

ii) We have $\quad E\theta_{ss}+n_1 = 0 \qquad \frac{\partial \underline{n}}{\partial s} = \underline{0}$ 1

$\therefore\ \underline{n} = \text{Const.} \qquad$ From $s=L$, $\underline{n} = \underline{F} = F\underline{e_1}$

$$\therefore\ n_1 = \underline{d_1}\cdot\underline{F} = F\underline{d_1}\cdot\underline{e_1} = -F\sin\theta$$

$$\therefore\ E\theta_{ss} - F\sin\theta = 0$$ 1

BCs $\quad \theta(s=0) = 0 \qquad \theta_s(s=L) = 0$ 1

iii) As above $\quad E\theta_{ss}+n_1 = 0, \quad n_2 = 0$

From above

$$\frac{\partial \underline{d_3}}{\partial s} = \theta_s \underline{d_1} \qquad \frac{\partial \underline{d_1}}{\partial s} = \theta_s(-\cos\theta, -\sin\theta, 0) = -\theta_s \underline{d_3}$$

From resistive force theory, the force on the filament per unit length is:

$$+\underline{f} = C_T U_T \underline{t} + C_N U_N \underline{n} = C_T U_T \underline{d_3} + C_N U_N \underline{d_1}$$

$$\therefore \underline{f} \cdot \underline{d_3} = f_3 = C_T U_T \qquad \underline{f} \cdot \underline{d_1} = f_1 = C_N U_N$$

$$= C_T \underline{e_2} \cdot \underline{d_3} U \qquad = C_N \underline{e_2} \cdot \underline{d_1} U$$

$$= C_T \sin\theta \, U \qquad = C_N \cos\theta \, U$$ 2

$$\therefore 0 = \underline{d_1}\left[\frac{\partial n_1}{\partial s} + \theta_s n_3 + f_1\right] + \underline{d_3}\left[\frac{\partial n_3}{\partial s} - \theta_s n_1 + f_3\right]$$ 1

$$\therefore 0 = E\theta_{ss} + n_1 = \frac{\partial n_1}{\partial s} + \theta_s n_3 + C_N \cos\theta U = \frac{\partial n_3}{\partial s} - \theta_s + C_T \sin\theta U$$ 1

$\underline{\text{BCs}}$ $\quad \theta(s=0) = 0 \qquad \theta_s(s=L) = 0$, as above

$\underline{n}(s=L) = F\underline{e_1} \qquad \therefore n_1(s=L) = -F\sin\theta$ 1

$n_3(s=L) = F\cos\theta$

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Q2

a) $dS = \left|\frac{\partial \underline{r}}{\partial \xi^1} \wedge \frac{\partial \underline{r}}{\partial \xi^2}\right| d\xi^1 d\xi^2$ $\qquad |\underline{a}\wedge\underline{b}|^2 = (\underline{a}\wedge\underline{b})\cdot(\underline{a}\wedge\underline{b})$ $= a^2 b^2 - (\underline{a}\cdot\underline{b})^2$ 2

$= \left[\left|\frac{\partial \underline{r}}{\partial \xi^1}\right|^2 \left(\frac{\partial \underline{r}}{\partial \xi^2}\right)^2 - \left(\frac{\partial \underline{r}}{\partial \xi^1}\cdot\frac{\partial \underline{r}}{\partial \xi^2}\right)^2\right]^{1/2} d\xi^1 d\xi^2$ 1

$= \left[g_{11}g_{22} - g_{12}^2\right]^{1/2} d\xi^1 d\xi^2 = \sqrt{\det g}\, d\xi^1 d\xi^2$ 1

b) $\underline{r} = R(1+\varepsilon h(\theta))\underline{e}_r + z\underline{e}_z$

$\frac{\partial \underline{r}}{\partial \theta} = R(1+\varepsilon h)\underline{e}_\theta + \varepsilon R h_\theta \underline{e}_r \qquad \frac{\partial \underline{r}}{\partial z} = \underline{e}_z$ 2

$\therefore\ g = \begin{pmatrix} R^2((1+\varepsilon h)^2 + \varepsilon^2 h_\theta^2) & 0 \\ 0 & 1 \end{pmatrix}$ 2

$\therefore\ \sqrt{\det g} = R\left[1 + 2\varepsilon h + \varepsilon^2 h^2 + \varepsilon^2 h_\theta^2\right]^{1/2} + O(\varepsilon^3)$

$= R\left[1 + \varepsilon h + \frac{\varepsilon^2 h^2}{2} + \frac{\varepsilon^2 h_\theta^2}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdot 4\varepsilon^2 h^2\right] + \ldots$

$= R\left[1 + \varepsilon h + \frac{\varepsilon^2 h_\theta^2}{2}\right] + O(\varepsilon^3)$ 2

c) i) h real $\therefore\ h_{-m} = h_m^*$. 1

For very large N, oscillations on a scale that is below the resolution of the continuum approximation. 1

ii) $E_M = \gamma \int dS = \gamma \underbrace{\int dz}_{\text{Unit Length}} \int d\theta\, R\left(1 + \varepsilon h + \frac{\varepsilon^2 h_\theta^2}{2}\right) + O(\varepsilon^3)$

$$\therefore E_M = \gamma R + \gamma R \varepsilon \underbrace{\int_0^{2\pi} d\theta \sum_{\substack{m=-N \\ m\neq 0}}^{N} h_m e^{im\theta}}_{0} + \frac{\gamma R \varepsilon^2}{2} \int_0^{2\pi} h_\theta^2 \, d\theta$$

$$= \gamma R + \frac{\gamma R \varepsilon^2}{2} \int_0^{2\pi} d\theta \sum_{m,n} h_m h_n e^{i(m+n)\theta} (-1)\cdot mn \qquad 2$$

$$\int_0^{2\pi} d\theta \, e^{i(m+n)\theta} = 2\pi \delta_{m,-n}$$

$$= \gamma R + \frac{\gamma R \varepsilon^2}{2} \sum_m h_m h_{-m} (-1)\cdot(-m^2) \qquad 2$$

$$= \gamma R + \frac{\gamma R \varepsilon^2}{2} \sum_m |h_m|^2 m^2 \quad \text{using } h_{-m} = h_m^*$$

iii) $$Z = (2\pi)^N \int_0^\infty da_1 \ldots da_N \underbrace{(a_1 \ldots a_N) e^{-\frac{\gamma R}{k_bT}} e^{-\frac{\gamma R \varepsilon^2}{2k_bT} \sum_m m^2 |h_m|^2}}$$

$$\therefore \langle |h_m|^2 \rangle = \langle a_m^2 \rangle = \frac{1}{Z} \cdot (2\pi)^{2N} \int_0^\infty da_1 \ldots da_N (a_m^2) \{ \quad \}$$

factor of 2 by counting -m term

$$= \frac{\int_0^\infty da_m \, a_m^3 e^{-\eta \varepsilon^2 a_m^2 \cdot m^2}}{\int_0^\infty da_m \, a_m e^{-\eta \varepsilon^2 a_m^2 m^2}} \quad \text{with } \eta = \frac{\gamma R}{k_bT}$$

$$\xi = \varepsilon^2 m^2 \eta = \frac{\gamma R}{k_bT} \varepsilon^2 m^2$$

$$= \frac{\int_0^\infty da_m \, a_m^3 e^{-\xi a_m^2}}{\int_0^\infty da_m \, a_m e^{-\xi a_m^2}} \qquad 3$$

$$= -\frac{\partial}{\partial \xi} \ln \int_0^\infty da_m \, a_m e^{-\xi a_m^2} = -\frac{\partial}{\partial \xi} \ln \left[\frac{-1}{2\xi} e^{-\xi a_m^2} \right]_0^\infty$$

$$= \frac{\partial}{\partial q} \ln q = \frac{1}{q} = \frac{k_b T}{\gamma R m^2 \varepsilon^2}.$$ 2

iv)

$$\therefore Z = \int da_1 \ldots da_N (a_{-N} \ldots a_N) \int_0^{2\pi} \mathrm{d}\psi_1 \ldots \int_0^{2\pi} \mathrm{d}\psi_N \exp\left[-\frac{\gamma R}{k_b T} - \frac{\gamma R \varepsilon^2}{2 k_b T} \sum_m m^2 |h_m|^2\right]$$

$$\underbrace{\exp\left[-\frac{\gamma R \varepsilon^3}{3 k_b T} \sum_{m,n} mn\, h_m h_n h_{-(n+m)}\right]}_{1 - \frac{\gamma R \varepsilon^3}{3 k_b T} \sum_{m,n} mn\, a_m a_n a_{-(n+m)} \cdot e^{i\psi_m} e^{i\psi_n} e^{i\psi_{-(m+n)}} + \text{h.o.t}}$$ 1

In each contribution to Z we have a term at $O(\varepsilon^3)$

$$\int d\psi_1 \ldots d\psi_N \underbrace{e^{i\psi_m} e^{i\psi_n} e^{i\psi_{-(m+n)}}}_{e^{i\psi_m} e^{i\psi_n} e^{-i\psi_{m+n}}}$$

① $m \neq \pm n$, $m \neq \pm(m+n)$ 1

ψ_m independent $\int d\psi_m e^{i\psi_m} = 0$, no contribution

② More generally even if the terms are dependent we have $\psi_m = \pm \psi_n = \pm \psi_{m+n}$.

No combination will give a zero exponent and thus the integral will always be zero 2

<u>Alternative</u> consider other possibilities

i) $m = \pm n$, $m \neq \pm(m+n)$

ii) $m \neq \pm n$, $m = \pm(m+n)$

iii) $m = \pm n = \pm(m+n)$

and show no non-zero contribution.

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Q3. a) i) PDE equations have no time dependence 1

ii) Equation of motion for Swimmer

$$M\frac{d^2\underline{X}_G}{dt^2} = \int_{\partial\Omega(t)} \sigma_{ij} n_j \, ds \, , \quad \partial\Omega(t) \text{ boundary of swimmer.}$$

Ratio of terms using $U \sim L/T$

$$\frac{(\rho L^3)\cdot U/T}{L^2 \mu U/L} \sim \frac{\rho}{\mu}\frac{L^3 U \cdot U/L}{UL} \sim \frac{\rho U L}{\mu} \sim Re$$

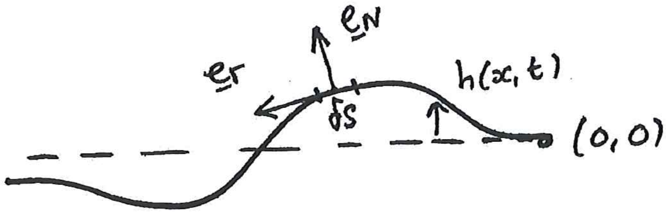
3

$\therefore$ Net force is zero as $Re \to 0$

b) i) $ds^2 = (1+\varepsilon^2 h_x^2)dx^2 \quad \therefore \quad L = \int ds = \int_{x^*}^{0} \sqrt{1+\varepsilon^2 h_x^2}\, dx$ 1

$$= -x^*(1+O(\varepsilon^2))$$

ii)



$$\delta s \underline{f}^{drag} = \delta s \left[-C_N U_N \underline{e}_N - C_T U_T \underline{e}_T \right]$$

~~3~~ 2

$\underline{e}_T = (-1, \varepsilon h_x)(1+O(\varepsilon^2)) \qquad \underline{U} = (0, \varepsilon h_t)$

$\underline{e}_N = (\varepsilon h_x, 1)(1+O(\varepsilon^2))$

$\therefore \; U_N = \underline{e}_N \cdot \underline{U} = \varepsilon h_t + O(\varepsilon^3)$ 3

$$u_T = \underline{e}_T \cdot \underline{u} = \varepsilon^2 h_x h_t + O(\varepsilon^4)$$ 1

$$\therefore \delta x \, \underline{f}^{drag} = -\delta x \left[C_N \begin{pmatrix} \varepsilon^2 h_x h_t \\ \varepsilon h_t \end{pmatrix} + C_T \begin{pmatrix} -\varepsilon^2 h_x h_t \\ 0 \end{pmatrix} + O(\varepsilon^3) \right] \left[1 + O(\varepsilon^2) \right]$$ 2

$$\therefore \text{Total drag force} = -\int_{\underbrace{-L}_{x^* = -L(1+O(\varepsilon^2))}}^{0} dx \begin{pmatrix} (C_N - C_T)\varepsilon^2 h_x h_t \\ C_N \varepsilon h_t \end{pmatrix} + O(\varepsilon^3)$$

Total force on flagellum is zero. $\therefore$ Force on flagellum due to optical trap is minus the drag force. 2

$$\therefore \text{Total force, } \underline{f} = \int_{-L}^{0} dx \begin{pmatrix} (C_N - C_T)\varepsilon^2 h_x h_t \\ C_N \varepsilon h_t \end{pmatrix} + O(\varepsilon^3)$$ 2

c) i)

$$\frac{\partial}{\partial t} f_i^M = -\frac{\partial}{\partial t} \int_{\partial\Omega(t)} \sigma_{ij}^M n_j \, dS \overset{\text{div theorem}}{=} + \frac{\partial}{\partial t} \int_{S_\infty} \sigma_{ij}^M \, dS$$ 2

$$\underset{\text{surface time independent}}{=} + \int_{S_\infty} \frac{\partial \sigma_{ij}^M}{\partial t} \, dS \underset{\text{div theorem, using } \nabla \cdot \frac{\partial \sigma_{ij}^M}{\partial t} = 0}{=} - \int_{\partial\Omega(t)} \left(\frac{\partial \sigma_{ij}^M}{\partial t} \right) n_j \, dS$$ 2

$$\therefore \left(\lambda \frac{\partial}{\partial t} + 1\right) f_i^M = - \int_{\partial\Omega(t)} \left(\underbrace{\left(\lambda \frac{\partial}{\partial t} + 1\right) \sigma_{ij}^M}_{\left[-\left(\lambda \frac{\partial}{\partial t} p^M + p^M\right)\delta_{ij} + \left(\lambda \frac{\partial \tau_{ij}}{\partial t} + \tau_{ij}\right)\right]} \right) n_j \, dS$$ 2

$$= - \int_{\partial\Omega(t)} -\left(p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) n_j \, dS$$

$$= - \int_{\partial\Omega(t)} \sigma_{ij} n_j \, dS = f_i$$ 1

$$\therefore \quad \lambda \frac{d\underline{f}^M}{dt} + \underline{f}^M = \underline{f}.$$ At $t=0$, $\underline{u} = p = 0$ $\therefore \underline{f}^M(t=0)$. 1