B8.5 Graph Theory **Problem Sheet 3** **9/11/2018**

1. Find $\chi'(K_n)$ for each $n \geqslant 2$. (You should get different behaviour for n even and n odd.)

2. (a) Let $r \geqslant 2$ and suppose that G is a connected r-regular graph with $\chi'(G) = r$. Show that G does not contain any bridges.

   (b) Construct a 3-regular graph with $\chi'(G) > 3$.

3. Let G be a 3-regular graph with $\chi'(G) = 3$, and suppose that there is a unique 3-edge colouring of G (up to permuting the colours). Prove that G has exactly 3 Hamilton cycles. Are there arbitrarily large graphs with this property?

4. Suppose that we 2-colour the edges of K_n, not necessarily properly. Show that there are monochromatic paths P_1 and P_2 such that $V(P_1) \cup V(P_2) = V(K_n)$.

5. Let G be a graph in which every vertex has even degree. Show that G can be written as the edge disjoint union of cycles, plus (possibly) some isolated vertices.

6. The *girth* $g(G)$ of a graph G is the length of a shortest cycle (or ∞ if G is a forest). Show that if G is a planar graph with girth $g < \infty$ then $e(G) \leqslant \frac{g}{g-2}(|G| - 2)$. Deduce that $K_{3,3}$ is not planar.

7. Show that every triangle-free planar graph is 4-colourable.

8. For which $n \geqslant 3$ does there exist a planar graph G with n vertices such that

   (a) $e(G) = 3n - 6$?

   (b) G is triangle-free and $e(G) = 2n - 4$?

9. Show that if G is a planar graph with $\delta(G) = 5$, then $|G| \geqslant 12$. Can we have equality?

10. A plane *triangulation* is a plane graph in which every face is a triangle. Given a plane triangulation with $n \geqslant 3$ vertices, show that we can add one vertex and three edges to form a triangulation with $n + 1$ vertices. Can every triangulation be formed in this way? What is the point of this question?

11. Show that in any network $(\overrightarrow{G}, s, t, c)$ there is a flow f of maximum value which is *acyclic*: there is no directed cycle $x_1 \to x_2 \cdots \to x_t \to x_1$ with strictly positive flow along each edge.

Optional bonus questions. These may not be covered in classes; MFoCS students should attempt them! Some of these are hard*!*

12. A graph G is *k-list colourable* if, whenever each vertex v is assigned a list $L(v)$ of at least k colours, it is possible to colour each vertex with a colour from its list so that adjacent vertices receive distinct colours.

    (a) For each k, construct a graph which is 2-colourable but not k-list colourable.

    (b)* Construct a planar graph which is not 4-list colourable.

13. (a) Prove that every (not necessarily proper) 2-colouring of the edges of K_{3n-1} contains n vertex-disjoint edges of the same colour.

    (b) Show that this does not hold for K_{3n-2}.

14.* For which n can you construct a planar graph G with $|G| = n$, $\delta(G) = 5$ and $\Delta(G) = 6$?

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk