**B8.5 Graph Theory** **Problem Sheet 4** **21/11/2018**

1. Prove a version of the max-flow min-cut theorem for a network with multiple sources $s_1, \ldots, s_k$ and sinks $t_1, \ldots, t_\ell$. (One was outlined in lectures.)

2. Deduce Menger's Theorem from the Max-Flow-Min-Cut Theorem. [Hint: use the vertex form. Show that an acyclic flow of value $k$ with $f(xy) \in \{0, 1\}$ for all edges corresponds to $k$ independent paths.]

3. (a) Is there a network $(\overrightarrow{G}, s, t, c)$ containing edges $xy$ and $yx$ such that there are maximum value flows $f_i$ with $f_1(xy) > 0$ and $f_2(yx) > 0$?

   (b) Formulate and answer a better version of Q3(a). [Hint overleaf.]

   (c) (Harder.) Formulate and answer an even better version of Q3(a).

4. Show that if $G$ is an $r$-regular bipartite graph with $2n$ vertices, where $r \geqslant 1$, then $G$ contains a matching of size $n$ (i.e., a matching with $n$ edges). Deduce that $\chi'(G) = r$.

5. Let $G$ be a bipartite graph with bipartition $(V_1, V_2)$.

   (a) For $0 \leqslant d \leqslant |V_1|$, show that $G$ contains a matching of size $|V_1| - d$ if and only if $|\Gamma(S)| \geqslant |S| - d$ for every $S \subseteq V_1$.

   (b) Show that $G$ contains a 1-to-$r$ matching (i.e., a 'pairing' of each $v \in V_1$ with a set $A_v \subseteq \Gamma(v)$ such that $|A_v| = r$ for all $v \in V_1$ and the sets $A_v$ are disjoint) if and only if $|\Gamma(S)| \geqslant r|S|$ for every $S \subseteq V_1$.

6. Show that if $e$ is an edge of a graph $G$ then $\kappa(G) - 1 \leqslant \kappa(G - e) \leqslant \kappa(G)$.

7. Let $U$ be a set of $k$ vertices in a $k$-connected graph $G$, and $x$ a vertex not in $U$. Show that $G$ contains $k$ paths from $x$ to vertices in $U$ meeting only at the vertex $x$ (a 'fan' from $x$ to $U$).

8. Prove that if $G$ is $k$-connected, where $k \geqslant 2$, then every set of $k$ vertices of $G$ lies on a cycle. Is the converse true?

9. Determine $\mathrm{ex}(n, K_{1,3})$ for every $n$. Describe the extremal graphs.

10. Determine $\mathrm{ex}(n, 2K_2)$ for every $n$. What are the extremal graphs? (Here $2K_2$ means a pair of vertex-disjoint edges.)

11. Suppose that $G$ is a graph with $n > r + 1$ vertices and $t_r(n) + 1$ edges.

    (a) Prove that for every $p$ with $r + 1 < p \leqslant n$ there is a subgraph $H$ of $G$ with $|H| = p$ and $e(H) \geqslant t_r(p) + 1$. [*Hint: Try to copy the proof of Turán's Theorem. You may wish to write* $n = qr + x$ *where* $0 \leqslant x < r$*, and consider the cases* $x = 0$*,* $x = 1$ *and* $x \geqslant 2$.]

    (b) Prove that $G$ contains two copies of $K_{r+1}$ with exactly $r$ common vertices.

**PTO**

*Optional bonus questions. These may not be covered in classes; MFoCS students should attempt them!*

12. Let $H$ be a graph, and define $c_n(H) := \mathrm{ex}(n, H)/\binom{n}{2}$. Prove that $c_n(H) \leqslant c_{n-1}(H)$, and show that $\lim_{n\to\infty} c_n(H)$ exists.

13. Let $k \geqslant 2$ and let $G$ be a graph with $|G| = n \geqslant 2k - 1$. Prove that if $e(G) \geqslant (2k - 3)(n - k + 1) + 1$ then $G$ contains a subgraph $H$ such that $H$ is $k$-connected.

14. *For those who did not do Part A Graph Theory:* We say that an $n$-by-$n$ matrix is *doubly stochastic* if all its entries are nonnegative and every row and every column sums to 1. A matrix is a *permutation matrix* if it is doubly stochastic and all entries are 0 or 1 (i.e., every row and column contains a single 1, and all other entries are 0).

    Prove that every doubly stochastic matrix is a convex combination of permutation matrices.

Small hint for Q3(b): The answer to Q3(a) is 'yes' but for a very silly, simple reason. Rule this out to get a better question.

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If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk