Information Theory **Sheet 1**

1. A deck of n cards in order $1, 2, \ldots, n$ is provided. One card is removed at random and placed at a random position in the deck. What is the entropy of the resulting deck of cards?

2. (**Pooling inequality**)Let $a, b \geq 0$ with $a + b > 0$. Show that
$$-(a+b)\log(a+b) \leq -a\log a - b\log b \leq -(a+b)\log\frac{a+b}{2}$$
and that the first inequality becomes an equality iff $ab = 0$, the second inequality becomes an equality iff $a = b$.

3. (**Log sum inequality**)Let $a_1, \ldots, a_n, b_1, \ldots, b_n \geq 0$. Show that
$$\sum_{i=1}^{n} a_i \log\frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i\right)\log\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$
with equality iff $\frac{a_i}{b_i}$ is constant.

4. Let X, Y, Z be discrete random variables. Prove or provide a counterexample to the following statements:
   (a) $H(X,Y) = H(X) + H(Y)$,
   (b) $H(Y|X) \geq H(Y|X,Z)$,
   (c) $H(X) = H(10X)$,

5. Does there exist a discrete random variable X with a distribution such that $H(X) = \infty$? If so, describe it as explicit as possible.

6. Consider a sequence $X_0, X_1, X_2, \ldots$ of identically distributed and independent random variables that take values in a finite set $\mathcal{X}$ with $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{1}{2}$. Let $N = \inf\{n \geq 1 : X_n = 1\}$.
   (a) Calculate $\mathbb{E}[N]$,
   (b) Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[\log N] \leq f(H(X_0))$.

7. Partition the interval $[0, 1]$ into n disjoint sub-intervals of length $p_1, \ldots, p_n$. Let $X_1, X_2, \ldots$ be iid random variables, uniformly distributed on $[0, 1]$, and $Z_m(i)$ be the number of the $X_1, \ldots, X_m$ that lie in the ith interval of the partition. Show that
$$R_m = \prod_{i=1}^{n} p_i^{Z_m(i)} \text{ fulfills } \frac{1}{m}\log R_m \to_{n\to\infty} \sum_{i=1}^{n} p_i \log p_i \text{ with probability 1.}$$

8. Let $\mathcal{X}$ be a finite set, f a real-valued function $f : \mathcal{X} \to \mathbb{R}$ and fix $\alpha \in \mathbb{R}$. We want to maximise the entropy $H(X)$ of a random variable X taking values in $\mathcal{X}$ subject to the constraint
$$\mathbb{E}[f(X)] \leq \alpha. \tag{1}$$
Therefore show that if U denotes a uniformly distributed random variable on $\mathcal{X}$, it holds that
   (a) if $\max_{x\in\mathcal{X}} f(x) \geq \alpha \geq \mathbb{E}[f(U)]$ the entropy (1) is maximised if X is uniformly distributed on $\mathcal{X}$.

(b) if f is non-constant and $\min_{x \in \mathcal{X}} f(x) \leq \alpha < \mathbb{E}[f(U)]$, the entropy (1) is maximised under the distribution

$$\mathbb{P}(X = x) = \frac{\exp(\lambda f(x))}{\sum_{x \in \mathcal{X}} \exp(\lambda f(x))} \text{ for } x \in \mathcal{X}$$

for $\lambda < 0$ choosen such that $\mathbb{E}[f(X)] = \alpha$.