B8.1 Martingales Through Measure Theory

Problem Sheet 0, MT 2016

Q1. What does it mean that μ is an outer measure on a measurable space $(\Omega, \mathcal{F})$? What does it mean that μ is a measure on $(\Omega, \mathcal{F})$?
Suppose μ is a measure on $(\Omega, \mathcal{F})$. Define μ^* by

$$\mu^*(E) = \inf\left\{\sum_{k=1}^{\infty} \mu(E_k) : E_k \in \mathcal{F} \text{ for all } k, \text{ and } \bigcup_{k=1}^{\infty} E_k \supset E\right\}$$

for every $E \subset \Omega$. Show that μ^* is an outer measure on $\mathscr{P}(\Omega)$ the σ-algebra of all subsets of Ω.

Q2. Let $(\mathbb{R}, \mathcal{M}_{\text{Leb}}, m)$ be the Lebesgue space. What does it mean that a function $f : \mathbb{R} \to [-\infty, \infty]$ is Lebesgue measurable, what does it mean that f is Borel measurable? What does it mean that f is a simple function (with respect to the σ-algebra $\mathcal{M}_{\text{Leb}}$)?

Suppose f is Lebesgue measurable. Describe the definition of the Lebesgue integral of f.

What does it mean that f is p-th Lebesgue integrable, and what is the L^p-norm $||f||_p$ of f? Here $p \geq 1$.

Suppose f is p-th Lebesgue integrable on $\mathbb{R}$, where $p \geq 1$, show that

$$m\{|f| \geq \lambda\} \leq \frac{1}{\lambda^p}||f||_p^p$$

for all $\lambda > 0$. Hence prove that if f is Lebesgue integrable, then f is finite almost everywhere.

Q3. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions. State

1) MCT, MCT series version, Fatou's Lemma and DCT (Lebesgue's dominated convergence theorem).

2) What does it mean that f_n converges to f in L^1?

3) Show that, if $f_n \to f$ almost everywhere, then $f_n \to f$ in L^1 if and only if $\int_{\mathbb{R}} |f_n| \, dm \to \int_{\mathbb{R}} |f| \, dm$ as $n \to \infty$.

Q4. Let $\rho \geq 0$ be a Lebesgue measurable function on $\mathbb{R}$. Define

$$\mu(E) = \int_E \rho dm$$

for every $E \in \mathcal{M}_{\text{Leb}}$. Prove μ is a measure on $\mathcal{M}_\rho$. If $\rho \geq 0$ is continuous, then μ is σ-finite. Under what kind of condition on ρ so that μ is a finite measure?