B8.1 Probability, Measure and Martingales

Problem Sheet 3, MT 2018

Q1. Suppose X, Y and Z are three independent real random variables. Prove that $X+Y$ and Z are independent.

Q2. Roll a fair die until we get a six. Let Y be the total number of rolls and X the number of 1's. Show that

$$\mathbb{E}\left[X|Y\right] = \frac{1}{5}(Y-1) \text{ and } \mathbb{E}\left[X^2|Y\right] = \frac{1}{25}\left(Y^2+2Y-3\right).$$

Hint. By definition $\mathbb{E}\left[X|Y\right] = \mathbb{E}\left[X|\sigma\left\{Y\right\}\right]$, which is different from the conditional expectation $\mathbb{E}\left[X|Y=y\right]$ used in Prelims Probability. The latter is defined to be

$$\mathbb{E}\left[X|Y=y\right] = \sum_x x\mathbb{P}\left[X=x|Y=y\right]$$

where the conditional probability

$$\mathbb{P}\left[X=x|Y=y\right] = \frac{\mathbb{P}\left[X=x, Y=y\right]}{\mathbb{P}\left[Y=y\right]}.$$

On the other hand, $\mathbb{E}\left[X|Y\right]$ is a random variable, measurable with respect to $\sigma\left\{Y\right\}$. The relationship between these two notions are given by the following formula [Sheet 2]

$$\mathbb{E}\left[X|Y\right] = \sum_y \mathbb{E}\left[X|Y=y\right] 1_{\{Y=y\}}.$$

Q3. Let $(X_n)_{n\geq 1}$ be a sequence of independent real valued random variables.

1) Suppose

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} = 1\right] > 0.$$

Show that

$$\frac{X_1+\cdots+X_n}{n} \to 1 \text{ with probability } 1.$$

2) Suppose

$$\mathbb{P}\left[X_n = n\right] = \mathbb{P}\left[X_n = -n\right] = \frac{1}{2(n+1)\log(n+1)}$$

and

$$\mathbb{P}\left[X_n = 0\right] = 1 - \frac{1}{(n+1)\log(n+1)}.$$

Let $S_n = \sum_{k=1}^n X_k$. Show that $\frac{S_n}{n} \to 0$ in probability, that is,

$$\mathbb{P}\left[\left|\frac{S_n}{n}\right| > \delta\right] \to 0$$

as $n \to \infty$ for every $\delta > 0$, but not almost surely.

[*Hint.* Calculate the variance of S_n to show the convergence in probability. Use the Borel-Cantelli lemmas to handle the almost sure convergence.]

Q4. 1) Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with $\mathbb{E}[X_n] = 1$ for all n. Let $\mathcal{F}_n = \sigma\{X_m : m \leq n\}$ and

$$M_0 = 1,\ M_n = \prod_{k=1}^{n} X_k$$

for $n = 1, 2, \cdots$. Show that $M = (M_n)_{n\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_n)$.

2) Let $(Y_n)_{n\geq 1}$ be a sequence of independent, identically distributed random variables such that the moment generating function $\psi(t) = \mathbb{E}\left[e^{tY_1}\right] < \infty$ for all t. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} Y_k$. Let

$$M_n = \frac{e^{tS_n}}{\psi(t)^n}\ ,\ n = 0, 1, 2, \cdots .$$

Show that $M = (M_n)_{n\geq 0}$ is a martingale (which is called an *exponential martingale*), so that it is a positive martingale.

Q5. Let $(\mathcal{F}_n)$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$.

1) Show that a random variable T taking values in $\mathbb{Z} \cup \{\infty\}$ is a stopping time w.r.t. $(\mathcal{F}_n)$ if and only if $\{T > n\} \in \mathcal{F}_n$ for every $n \in \mathbb{Z}$.

2) Let S and T be stopping times w.r.t. $(\mathcal{F}_n)$. Prove that $S+T$, $\max\{S,T\}$ and $\min\{S,T\}$ are stopping times too. Assuming it is always non-negative, must $S - T$ be a stopping time?

Q6. Let $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ be a filtered probability space. Suppose $Y = (Y_n)_{n\geq 1}$ is a submartingale and $V = (V_n)_{n\geq 0}$ is predictable, and $V_n \geq 0$ for all n. Show that (subject to integrability) $X_n = \sum_{k=1}^{n} V_k(Y_k - Y_{k-1})$, $Y_0 = 0$, is a submartingale.