B8.1 Probability, Measure and Martingales

Problem Sheet 4, MT 2018

Q1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
a) Suppose $\{\mathcal{G}_\alpha : \alpha \in \Lambda\}$ is a family of sub σ-algebras of $\mathcal{F}$, and ξ is integrable. Let $X_\alpha = \mathbb{E}[\xi|\mathcal{G}_\alpha]$ for $\alpha \in \Lambda$. Show that $\{X_\alpha : \alpha \in \Lambda\}$ is uniformly integrable.
b) Suppose $\{Y_\alpha : \alpha \in \Lambda\}$ is a bounded subset of $L^p(\Omega, \mathcal{F}, \mathbb{P})$, where $p > 1$, show that $\{Y_\alpha : \alpha \in \Lambda\}$ is uniformly integrable.

Q2. Let $X = (X)_{n\geq 0}$ be a martingale and T be a *finite* stopping time. Suppose X_T is integrable. Show that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ if and only if

$$\lim_{n\to\infty} \mathbb{E}[X_n : T > n] = 0.$$

Hint. Note that since T is finite so that $X_{T\wedge n}$ tends X_T as $n \to \infty$. Therefore Theorem 10.4 may be useful.

Q3. Let $X = (X_n)_{n\geq 0}$ be a martingale such that $\sup_{n\geq 0} \mathbb{E}[X_n^2] < \infty$. Show that X_n converges to a random variable X_∞ with probability 1 and in L^2 as well.

Q4. Let $(Z_n)_{n\geq 0}$ be an adapted random sequence on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ such that each Z_n is integrable. Let N be a positive integer. Define recursively and backwards by

$$X_N = Z_N$$

and

$$X_n = \max\{Z_n, \mathbb{E}[X_{n+1}|\mathcal{F}_n]\}$$

for $n = N-1, \cdots, 1, 0$. Show that $(X_n)_{0\leq n\leq N}$ is a supermartingale, and that $X_n \geq Z_n$ for $n = 0, \cdots, N$. [(X_n) is called the Snell envelope of (Z_n).]

Q5. a) By considering $h(t) = \ln t - \frac{t}{e}$ for $t > 0$ or otherwise, show that $\ln t \leq t/e$ for $t > 0$.
b) Define $\log^+(t) = \max\{0, \ln t\}$ for $t > 0$, that is, $\log^+ t = 1_{(1,\infty)}(t) \ln t$ for $t > 0$. Show that

$$a \log^+ b \leq a \log^+ a + \frac{b}{e}$$

for any $a, b > 0$.
c) $\rho(t) = (t-1)^+$ for all $t \in \mathbb{R}$ is continuous and increasing, with right derivative

$$\rho'_+(t) = \rho'(t+) = 1_{[1,\infty)}(t).$$

Show that the Lebesgue-Stieltjes measure $m_\rho(dt) = 1_{[1,\infty)}(t)dt$. [*Hint. you may use the results in Q5 in Sheet 1.*]
d) Suppose $X = (X_n)$ is a non-negative submartingale, and $X_n^* = \sup_{k\leq n} X_k$ for every n. Show that, by using Doob's maximal inequality or otherwise,

$$\mathbb{E}[\rho(X_n^*)] \leq \mathbb{E}\left[X_n \int_{(0,X_n^*]} \frac{1}{t} m_\rho(dt)\right]$$

and conclude that

$$\mathbb{E}\left[X_n^*\right] \leq \frac{e}{e-1}\left(1+\mathbb{E}\left[X_n \log^+ X_n\right]\right)$$

for every n. [This is called Doob's L^1-inequality.]

Q6. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with the same distribution given by

$$\mathbb{P}\left[X_n = 1\right] = \mathbb{P}\left[X_n = -1\right] = \frac{1}{2}.$$

Define S_0 and $S_n = \sum_{k=1}^{n} X_k$ for $n = 1, 2, \cdots$. Show that $S = (S_n)$ is a martingale [you should specify a filtration]. Let $\psi(t) = \mathbb{E}\left[e^{tX_1}\right]$. Show that $\psi(t) = \frac{e^t + e^{-t}}{2}$.
a) Show that $M_n^{(t)} = e^{tS_n}\psi(t)^{-n}$ is a martingale [*Hint.* Question 4, Sheet 3]. Let $T = \inf\{n \geq 0 : S_n = 1\}$. Show that

$$\mathbb{E}\left[e^{tS_{T\wedge n}}\psi(t)^{-(T\wedge n)}\right] = 1$$

for every $n = 1, 2, \cdots$, and for every t.
b) Let $t > 0$. Show that $\left(e^{tS_{T\wedge n}}\right)_{n\geq 0}$ is bounded by e^t,

$$\lim_{n\to\infty} M_{T\wedge n}^{(t)} = M_T^{(t)} 1_{\{T<\infty\}}$$

and that

$$\mathbb{E}\left[M_T^{(t)} 1_{\{T<\infty\}}\right] = 1$$

for every $t > 0$.
c) Deduce that, from b), for $t > 0$,

$$\mathbb{E}\left[\psi(t)^{-T} 1_{\{T<\infty\}}\right] = e^{-t}$$

for all $t > 0$, and deduce that (by taking limit as $t \downarrow 0$ both sides, and using MCT)

$$\mathbb{P}\left[T < \infty\right] = 1.$$

d) Therefore, by definition of T, show that $S_T = 1$ with probability 1, but $\mathbb{E}\left[S_T\right] \neq \mathbb{E}\left[S_0\right]$.

Q7. Let $(X_n)_{n\geq 1}$ be a sequence of independent real random variables, such that $\mathbb{E}\left[X_n\right] = 0$ for every n and $\sum_{n=1}^{\infty} \mathbb{E}\left[X_n^2\right] < \infty$. Show that $\sum_{n=1}^{\infty} X_n$ converges with probability one.