B8.1 Probability, Measure and Martingales

Problem Sheet 1, 2018 MT

Q1. (*Monotone class theorem for functions*) Let $\mathscr{C}$ be a π-system on Ω, and let $\mathscr{H}$ be a vector space of real-valued functions on Ω. Suppose

1) the constant functions belong to $\mathscr{H}$, and $1_E \in \mathscr{H}$ for every $E \in \mathscr{C}$, and

2) if $\{f_n\} \subseteq \mathscr{H}$ is an increasing sequence such that $f = \lim_{n\to\infty} f_n$ is finite, then $f \in \mathscr{H}$.

Prove that $\mathscr{H}$ contains all finite real-valued functions which are $\sigma(\mathscr{C})$-measurable, where $\sigma(\mathscr{C})$ is the smallest σ-algebra containing $\mathscr{C}$.

[*Hint.* Show that $\mathscr{D} = \{E \subset \Omega : 1_E \in \mathscr{H}\}$ is a monotone class and conclude that $\sigma(\mathscr{C}) \subset \mathscr{D}$. Then prove that any simple $\sigma(\mathscr{C})$-measurable function belongs to $\mathscr{H}$, and conclude your argument by using the structure theorem for measurable functions in terms of simple functions, see *item 7, on page 4, Lecture Notes*].

Q2. 1) We say $f : \mathbb{R}^n \to \mathbb{R}^m$ is Borel measurable, if $f^{-1}(G) \in \mathscr{B}(\mathbb{R}^n)$ whenever $G \in \mathscr{B}(\mathbb{R}^m)$, where $\mathscr{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra, i.e. the smallest σ-algebra containing all open subsets of $\mathbb{R}^n$. Prove that, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, then f is Borel measurable.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable, and define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x - y)$.

Show that F is Borel measurable on $\mathbb{R}^2$.

3) Let m be the Lebesgue measure on $\mathbb{R}$. Suppose f and g are Borel measurable and integrable on $\mathbb{R}$ with respect to m. Show that $f(x-y)g(y)$ is Borel measurable on $\mathbb{R}^2$ and show that

$$\int_{\mathbb{R}^2} |f(x-y)g(y)|\, dxdy = \|f\|_{L^1} \|g\|_{L^1}$$

where $dxdy$ is the Lebesgue measure on $\mathbb{R}^2$ and

$$\|f\|_{L^1} = \int_{\mathbb{R}} |f| dm$$

is the L^1-norm on the measure space $(\mathbb{R}, \mathcal{M}_{\text{Leb}}, m)$. Hence deduce that if f, g are Borel measurable and Lebesgue integrable, then the convolution $(f \star g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ is Lebesgue integrable and

$$||f \star g||_{L^1} \le ||f||_{L^1}||g||_{L^1}.$$

[*You may use the Fubini Theorem in an appropriate setting*].

Q3. Let (S, Σ) be a measurable space and $X_k : \Omega \to S$ be mappings, where $k = 1, 2, \cdots, n$. By definition, $\mathcal{G} = \sigma\{X_k : 1 \le k \le n\}$ is the smallest σ-algebra on Ω such that X_k are measurable mappings from $(\Omega, \mathcal{G})$ to (S, Σ).

1) Show that

$$\mathcal{G} = \sigma\left\{X_1^{-1}(A_1) \bigcap \cdots \bigcap X_n^{-1}(A_n) : A_k \in \Sigma \text{ for } 1 \le k \le n\right\}.$$

2) Suppose $Y : \Omega \to \mathbb{R}$. Then Y is $\mathcal{G}$-measurable if and only if $Y = F(X_1, \cdots, X_n)$ where $F : \prod_{k=1}^n S \to \mathbb{R}$ is $\prod_{k=1}^n \Sigma$-measurable. [*Here for product σ-algebra $\prod_{k=1}^n \Sigma$, see item 1, page 24 in Lecture Notes.*]

[*Hint.* Let $\mathscr{H}$ be the family of functions with form $f(X_1, \cdots, X_n)$. Apply the monotone class theorem (Q1) to $\mathscr{H}$].

3) If $(\Omega, \mathcal{F})$ is a measurable space and X_k $(k = 1, 2, \cdots, n)$ are n real-valued random variables on $(\Omega, \mathcal{F})$, then $Y : \Omega \to \mathbb{R}$ is $\sigma\{X_k : 1 \leq k \leq n\}$-measurable if and only if $Y = F(X_1, \cdots, X_n)$ for some $F : \mathbb{R}^n \to \mathbb{R}$ which is Borel measurable.

Q4. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let μ^* be the outer measure

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : \text{ where } A_i \in \mathcal{F} \text{ s.t. } \bigcup_{i=1}^{\infty} A_i \supset E\right\}$$

where $E \subset \Omega$. Let $\mathcal{F}^*$ be the σ-algebra of all μ^*-measurable subsets. Thus $(\Omega, \mathcal{F}^*, \mu^*)$ is a measure space, $\mathcal{F} \subset \mathcal{F}^*$ and $\mu^* = \mu$ on $\mathcal{F}$, so that μ^* will be denoted by μ for simplicity. [*Theorem 2.4, page 11 in Lecture Notes*].

1) If $E \in \mathcal{F}^*$, then there is a subset $B \in \mathcal{F}$ such that $E \subset B$ and $\mu^*(B \setminus E) = 0$. Hence conclude that $\mathcal{F}^* = \mathcal{F}^\mu$, where $\mathcal{F}^\mu = \sigma\{\mathcal{F}, \mathcal{N}\}$, $\mathcal{N}$ is the collection of all μ^*-null subsets.

[*Hint.* First consider the case that $\mu(E) < \infty$, so by definition, for every $N = 1, 2, \cdots$, there is a countable cover $A_i^{(N)}$ of E, $A_i^{(N)} \in \mathcal{F}$, such that

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(A_i^{(N)}) < \mu(E) + \frac{1}{2^N}.$$

Prove that $B = \bigcap_{N=1}^{\infty} \bigcup A_i^{(N)}$ is what you want].

2) Let ρ be an increasing function on $\mathbb{R}$, m_ρ be the Lebesgue-Stieltjes measure on the σ-algebra $\mathcal{M}_\rho$. We know that $\mathscr{B}(\mathbb{R}) \subset \mathcal{M}_\rho$ [*Section 3, Lecture Notes*], so that $(\mathbb{R}, \mathscr{B}(\mathbb{R}), m_\rho)$ is a measure space. Show that for every $E \in \mathcal{M}_\rho$ there is a Borel subset $B \in \mathscr{B}(\mathbb{R})$ such that $E \subset B$, and $m_\rho(B \setminus E) = 0$.

[*Hint.* Show that m_ρ is σ-finite, and apply 1) to $\mathcal{F} = \mathscr{B}(\mathbb{R})$.]

Q5. 1) Let $\rho(t) = t + 1$ for $t \geq 1$ and $\rho(t) = 0$ for $t < 1$. Prove that ρ is increasing, right continuous on $(-\infty, \infty)$. Calculate $m_\rho(A)$ where $A \subset (-\infty, 1)$, $m_\rho(\{1\})$ and $m_\rho(A)$ for $A \subset (1, \infty)$, where A is Borel measurable. Hence describe the Lebesgue-Stietljes measure m_ρ in terms of Lebesgue measure (and integrals).

2) ρ as in 1). Show that the right derivative $\rho'(t+)$ exists for all t, and is non-negative, hence it is Borel measurable. Define $\mu(A) = \int_A \rho'(t+)dt$ [where dt denotes the Lebesgue measure] for $A \in \mathscr{B}(\mathbb{R})$. Prove μ is a measure on $\mathscr{B}(\mathbb{R})$. Calculate $\mu(A)$ for $A \subset (-\infty, 1)$ and $A \subset (1, \infty)$, and calculate $\mu(\{1\})$ in terms of Lebesgue measure m. Conclude that $\mu \neq m_\rho$ on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

3) Suppose ρ is a *continuous increasing* function on $(-\infty, \infty)$, which is piece-wise differentiable in the sense that there are finite many $a_1 < a_2 < \cdots < a_n$ such that ρ has *continuous* derivative on (a_i, a_{i+1}) for $i = 0, \cdots, n$ (with $a_0 = -\infty$ and $a_{n+1} = \infty$), [Examples including such as (1) $\rho(t) = t - 1$ for $t \geq 1$ and $\rho(t) = 0$ for $t < 0$; (2) $\rho(t) = t^p$ for $t > 0$ $\rho(t) = 0$ for $t \leq 0$ where $p > 1$ a constant].

In particular the derivative ρ' is non-negative, continuous except for finite many points, thus must be Borel measurable. Let $\mu(A) = \int_A \rho'(t)dt$ and m_ρ denote the associated Lebesgue-Stietljes measure. Prove that $\mu = m_\rho$ on $\mathscr{B}(\mathbb{R})$.

[*Hint.* Show that $\mu = m_\rho$ on the π-system $\mathscr{C}$ of all $(s, t]$, by using the Fundamental Theorem in Calculus].