B8.1 Probability, Measure and Martingales

Problem Sheet 2, 2018 MT

Q1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{G}$ be a sub σ-algebra. If X is integrable, then $\mathbb{E}[X|\mathcal{G}]$ is the conditional expectation of X given $\mathcal{G}$, as defined in the lectures, so that $\mathbb{E}[X|\mathcal{G}]$ is the unique Y (up to equal almost surely) which is $\mathcal{G}$-measurable and integrable, such that

$$\mathbb{E}[X : A] = \mathbb{E}[Y : A] \text{ for every } A \in \mathcal{G}.$$

1) Give a definition for the conditional expectation $\mathbb{E}[X|\mathcal{G}]$ when X is a non-negative random variable.

2) Show that $\mathbb{E}[X_1|\mathcal{G}] \geq \mathbb{E}[X_2|\mathcal{G}]$ almost surely if X_1, X_2 are integrable and $X_1 \geq X_2$ almost surely.

Q2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{G}_1, \mathcal{G}_2$ be two independent sub σ-algebras of $\mathcal{F}$. Suppose X is an integrable random variable, and suppose $\{X, \mathcal{G}_1\}$ and $\mathcal{G}_2$ are independent [by definition, it means that the σ-algebra $\sigma\{X, \mathcal{G}_1\}$ and $\mathcal{G}_2$ are independent]. Show that

$$\mathbb{E}[X|\mathcal{G}_1, \mathcal{G}_2] = \mathbb{E}[X|\mathcal{G}_1],$$

where $\mathbb{E}[X|\mathcal{G}_1, \mathcal{G}_2]$ denotes the conditional expectation of X given $\sigma\{\mathcal{G}_1 \cup \mathcal{G}_2\}$.

Hence deduce that if $\{X_\alpha\}$ and $\{Y_\beta\}$ are two independent families of random variables [by definition, it means that $\sigma\{X_\alpha\}$ and $\sigma\{Y_\beta\}$ are independent], and suppose $\{X, X_\alpha\}$ and $\{Y_\beta\}$ are independent, then

$$\mathbb{E}[X|\{X_\alpha\}, \{Y_\beta\}] = \mathbb{E}[X|\{X_\alpha\}].$$

Q3. 1) Let X and $X_1, \cdots, X_n$ be (real-valued) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose X is integrable. Show that there is a Borel measurable function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbb{E}[X|X_1, \cdots, X_n] = f(X_1, \cdots, X_n).$$

[*Hint.* You may use the results in Q3 in Problem Sheet 1.]

2) Suppose X and Y are two independent integrable random variables with the same distribution. Calculate

(a) $\mathbb{E}[X|X, Y]$ and $\mathbb{E}[X|Y]$;

(b) $\mathbb{E}[X|X+Y]$, $\mathbb{E}[Y|X+Y]$;

(c) $\mathbb{E}[h(X,Y)|X+Y, X-Y]$ for a Borel measurable function so that $h(X,Y)$ is integrable.

Q4. 1) Let X and Y be discrete random variables taking values in $\{0, 1, \cdots, N\}$. Suppose the joint distribution of (X, Y) is given by

$$\mathbb{P}[X = i, Y = j] = q_{ij} > 0$$

for all i, j. Derive a formula for $\mathbb{E}[f(X)|Y]$ where f is a function on $\{0, 1, \cdots, N\}$, in terms of q_{ij}. Verify that $\mathbb{E}[f(X)|Y] = \mathbb{E}[f(X)]$ if X and Y are independent.

2) Let $\{A_n : n = 1, 2, \cdots\}$ be a countable partition of the sample space Ω, where $A_n \in \mathcal{F}$ and $\bigcup_{n=1}^{\infty} A_n = \Omega$ with $\mathbb{P}(A_n) > 0$ for every n. Let X be an integrable variable. Show that

$$\mathbb{E}[X|\sigma\{A_n : n \geq 1\}] = \sum_{n=1}^{\infty} \frac{\mathbb{E}[X 1_{A_n}]}{\mathbb{P}(A_n)} 1_{A_n}.$$

3) Suppose $X_1, \cdots, X_n$ are random variables taking values in a finite or countable set, and Z is another random variable. Prove that

$$\mathbb{E}[Z|X_1,\cdots,X_n] = f(X_1,\cdots,X_n)$$

where

$$f(x_1,\cdots,x_n) = \mathbb{E}[Z|X_1 = x_1,\cdots,X_n = x_n].$$

Q5. Suppose $\{X_n : n = 1, 2, \cdots\}$ is a sequence of independent and identically distributed random variables, and

$$\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = \frac{1}{2}.$$

For every $z \in \mathbb{Z}$ define

$$A_z = \{S_n = z \text{ for infinitely many } n\},$$

$$B_- = \left\{\liminf_{n\to\infty} S_n = -\infty\right\} \text{ and } B_+ = \left\{\limsup_{n\to\infty} S_n = \infty\right\}.$$

1) Let $\mathcal{G}_\infty = \bigcap_{n=1}^\infty \sigma\{X_k : k \geq n\}$ be the tail σ-algebra of $\{X_k : k \geq 1\}$. Show that both B_- and B_+ are tail events, i.e. belong to $\mathcal{G}_\infty$, and show that $\mathbb{P}[B_\pm] = 0$ or 1. Show that $\mathbb{P}[B_-] = \mathbb{P}[B_+]$.

2) Using the Borel-Cantelli lemma to show that, for all $k \geq 1$

$$\limsup_{n\to\infty} (S_{n+k} - S_n) = k \text{ almost surely.}$$

[*Hint.* Let $A_n = \{S_{n+k} - S_n = k\}$, and compute $\mathbb{P}[A_n]$.]

3) Deduce that $\mathbb{P}\left[B_-^c \cap B_+^c\right] = 0$, and therefore $\mathbb{P}[B_-] = \mathbb{P}[B_+] = 1$. Hence conclude that $\mathbb{P}[A_z] = 1$ for every $z \in \mathbb{Z}$.

Q6. 1) Let Z be a random variable taking its values in $\mathbb{Z}_+$. Show that $\mathbb{E}[Z] = \sum_{n=1}^\infty \mathbb{P}[Z \geq n]$.

2) Let X be an integrable random variable. Show that

$$\sum_{n=1}^\infty \mathbb{P}[|X| \geq \varepsilon n] < \infty.$$

[*Hint.* Consider $Z = [|X|/\varepsilon]$ the integer part of $|X|/\varepsilon$ which is integer valued, so 1) is applicable to Z.]

3) Let $\{X_n : n = 1, 2, \cdots\}$ be a sequence of independent and identically distributed random variables and $M_n = \sup_{k\leq n} |X_k|$ for $n = 1, 2, \cdots$. Suppose X_1 is p-th integrable, i.e. $\mathbb{E}[|X_1|^p] < \infty$ for some $p \in (0, \infty)$. Show that

$$\frac{1}{n^{1/p}} M_n \to 0$$

with probability 1.

[*Hint.* Given any $\varepsilon > 0$, consider $A_n = \left\{|X_n| \geq \varepsilon n^{1/p}\right\}$ and apply the Borel-Cantelli lemma.]