B8.1 Martingales Through Measure Theory

Problem Sheet 0, solutions, MT 2016

Q1. μ is an outer measure on a measurable space $(\Omega, \mathcal{F})$ if (1) $\mu(\emptyset) = 0$, (2) $\mu(A) \leq \mu(B)$ if $A, B \in \mathcal{F}$ and $A \subset B$, (3) μ is countably sub-additive:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

for any sequence $A_n \in \mathcal{F}$.
μ is a measure on $(\Omega, \mathcal{F})$ if (1) and (2) above are satisfied, and (3)' μ is countably additive:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

for any *disjoint* sequence $A_n \in \mathcal{F}$.

Let us prove μ^* is an outer measure. Let $A_n \in \mathcal{F}$. If $\sum_{n=1}^{\infty} \mu^*(A_n) = \infty$ then the sub-additivity is trivial, thus we assume that $\sum_{n=1}^{\infty} \mu^*(A_n) < \infty$. Hence $\mu^*(A_n) < \infty$ for each n. By definition, for every $\varepsilon > 0$ there is a countable cover $\{E_i^{(n)} : i = 1, 2, \cdots\}$ of A_n, where $E_i^{(n)} \in \mathcal{F}$, such that $\sum_{i=1}^{\infty} |E_i^{(n)}| \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$. Then $\{E_i^{(n)} : i, n = 1, 2, \cdots\}$ forms a countable cover of $\cup_{n=1}^{\infty} A_n$ so that

$$\begin{aligned}\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \mu\left(E_i^{(n)}\right) \leq \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) \\ &= \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, so by letting $\varepsilon \downarrow 0$ we conclude that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

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Q2. Let $(\mathbb{R}, \mathcal{M}_{\text{Leb}}, m)$ be the Lebesgue space. A function $f : \mathbb{R} \to [-\infty, \infty]$ is Lebesgue (resp. Borel) measurable if for every Borel subset G, $f^{-1}(G)$ is Lebesgue (resp. Borel) measurable and both $f^{-1}(\infty)$ and $f^{-1}(-\infty)$ are Lebesgue (resp. Borel) measurable.
f is a simple function (with respect to the σ-algebra $\mathcal{M}_{\text{Leb}}$) if $f = \sum_{k=1}^{n} c_k 1_{E_k}$ for some n,some real numbers c_k and some Lenesgue measurable subsets E_k.

If φ is non-negative, Lebesgue measureable simple function, and if $\varphi = \sum_{k=1}^{n} c_k 1_{E_k}$ where $c_k \geq 0$ are real numbers, E_k are Lebesgue measurable, then

$$\int_{\mathbb{R}} \varphi dm = \sum_{k=1}^{n} c_k m(E_k).$$

If $f \geq 0$ is Lebesgue measurable we define

$$\int_{\mathbb{R}} f dm = \sup\left\{\int_{\mathbb{R}} \varphi dm : \varphi \geq 0 \text{ simple, and } \varphi \leq f\right\}.$$

In this case, i.e. for a non-negative measurable function f, we say f is integrable if $\int_{\mathbb{R}} f dm < \infty$. If f is measurable, then both $f^+ = \max\{f, 0\}$ and $f^- \max\{-f, 0\}$ are measurable, f is integrable if both f^+ and f^- are integrable, and define

$$\int_{\mathbb{R}} f dm = \int_{\mathbb{R}} f^+ dm - \int_{\mathbb{R}} f^- dm.$$

f is p-th Lebesgue integrable if f is measuirable, and if $|f|^p$ is integrable. The L^p-norm $||f||_p = \left(\int_{\mathbb{R}} |f|^p dm\right)^{\frac{1}{p}}$.
Suppose f is p-th Lebesgue integrable on $\mathbb{R}$, where $p \geq 1$, then

$$m\{|f| \geq \lambda\} = \int_{\{|f|\geq\lambda\}} dm \leq \int_{\mathbb{R}} \frac{|f|^p}{\lambda^p} dm = \frac{1}{\lambda^p} \|f\|_p^p .$$

In particular, if f is Lebesgue integrable, then

$$m\{|f| \geq \lambda\} \leq \frac{1}{\lambda} \int_{\mathbb{R}} |f| dm.$$

While

$$m\{|f| = \infty\} \leq m\{|f| \geq \lambda\} \leq \frac{1}{\lambda} \int_{\mathbb{R}} |f| dm$$

for any $\lambda > 0$, by letting $\lambda \uparrow \infty$ to obtain

$$m\{|f| = \infty\} \leq \lim_{\lambda\uparrow\infty} \frac{1}{\lambda} \int_{\mathbb{R}} |f| dm = 0$$

so that $\{|f| = \infty\}$ is a Lebesgue zero set, so f is finite almost everywhere.

Q3. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions. State
1) MCT – If $f_{n+1} \geq f_n \geq 0$ almost everywhere for every n, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n dm = \int_{\mathbb{R}} f dm.$$

where $f = \lim_{n\to\infty} f_n$.
MCT series version – if $f_n \geq 0$ almost everywhere for all n then

$$\int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n dm = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n dm.$$

Fatou's Lemma – if $f_n \geq 0$ for all n, then

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n dm \leq \liminf_{n\to\infty} \int_{\mathbb{R}} f_n dm.$$

DCT (Lebesgue's dominated convergence theorem) –If $f_n \to f$ almost everywhere, and there is an integrable function g such that $|f_n| \leq g$ almost everywhere for all n, then all f_n and f are integrable and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n dm = \int_{\mathbb{R}} f dm.$$

2) f_n converges to f in L^1 if $\int_{\mathbb{R}} |f_n - f| \, dm \to 0$ as $n \to \infty$.
3) Show that, if $f_n \to f$ almost everywhere, then $f_n \to f$ in L^1 if and only if $\int_{\mathbb{R}} |f_n| \, dm \to \int_{\mathbb{R}} |f| \, dm$ as $n \to \infty$.
"Only if" part is easy. In fact, if $f_n \to f$ in L^1, then, by the triangle inequality,

$$||f_n| - |f|| \leq |f_n - f|$$

so that

$$0 \leq \left| \int_{\mathbb{R}} |f_n| dm - \int_{\mathbb{R}} |f| dm \right| = \left| \int_{\mathbb{R}} (|f_n| - |f|) \right| \leq \int_{\mathbb{R}} |f_n - f| d\mu \to 0$$

which implies that $\int_{\mathbb{R}} |f_n| d\mu \to \int_{\mathbb{R}} |f| d\mu$.
Proof of "If" part. Assume that $f_n \to f$ almost surely and $\int_{\mathbb{R}} |f_n| d\mu \to \int_{\mathbb{R}} |f| d\mu$, show that $f_n \to f$ in L^1. Let $A_n = \{f_n f \geq 0\}$, $B_n = \{f_n f < 0\}$. Then

$$|f_n - f| = ||f_n| - |f|| \quad \text{ on } A_n$$

and, by the triangle inequality,

$$|f_n - f| = ||f_n| + |f|| \leq ||f_n| - |f|| + 2|f| \quad \text{on } B_n.$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}} |f_n - f| d\mu &= \int_{A_n} |f_n - f| \, d\mu + \int_{B_n} |f_n - f| \, d\mu \\
&\leq \int_{A_n} ||f_n| - |f|| \, d\mu + \int_{B_n} [||f_n| - |f|| + 2|f|] \, d\mu \\
&= \int_{\mathbb{R}} ||f_n| - |f|| \, d\mu + 2 \int_{B_n} |f| d\mu \\
&= \int_{\mathbb{R}} ||f_n| - |f|| \, d\mu + 2 \int_{\mathbb{R}} 1_{B_n} |f| d\mu.
\end{aligned}$$

The first term on the right-hand side of the previous inequality many be rewrriten as the following

$$\begin{aligned}
\int_{\mathbb{R}} ||f_n| - |f|| \, d\mu &= \int_{\mathbb{R}} (|f_n| - |f|)^+ \, d\mu + \int_{\mathbb{R}} (|f_n| - |f|)^- \, d\mu \\
&= \int_{\mathbb{R}} (|f_n| - |f|) \, d\mu + 2 \int_{\mathbb{R}} (|f_n| - |f|)^- \, d\mu
\end{aligned}$$

where we have used the identity

$$|g| = g^+ + g^- = g^+ - g^- + 2g^- = g + 2g^-.$$

Putting together we obtain the following estimate for the L^1-norm of $f_n - f$:

$$\begin{aligned}
\int_{\mathbb{R}} |f_n - f| d\mu &\leq \int_{\mathbb{R}} ||f_n| - |f|| \, d\mu + 2 \int_{\mathbb{R}} 1_{B_n} |f| d\mu \\
&= \int_{\mathbb{R}} (|f_n| - |f|) \, d\mu + 2 \int_{\mathbb{R}} (|f_n| - |f|)^- \, d\mu + 2 \int_{\mathbb{R}} 1_{B_n} |f| d\mu. \qquad (1)
\end{aligned}$$

We next want to let $n \to \infty$ in the inequality above. The first term on the right-hand side tends to zero as $n \to \infty$ by assumption. In fact

$$\int_{\mathbb{R}} (|f_n| - |f|) \, d\mu = \int_{\mathbb{R}} |f_n| d\mu - \int_{\mathbb{R}} |f| d\mu \to 0$$

as $n \to \infty$. For the second term, we observe that

$$(|f_n| - |f|)^- = 0 \qquad \text{on } \{|f_n| \geq |f|\}$$

and

$$(|f_n| - |f|)^- = |f_n| - |f| \leq |f_n| \leq |f| \qquad \text{on } \{|f_n| < |f|\}$$

so that

$$(|f_n| - |f|)^- \leq |f|$$

for all n, $|f|$ is integrable, and $(|f_n| - |f|)^- \to 0$ almost surely, thus by the Dominated Convergence Theorem we conclude that

$$\int_{\mathbb{R}} (|f_n| - |f|)^- \, d\mu \to 0.$$

To show the last term on the right-hand side of (1) $\int_{B_n} |f| d\mu$ tends to zero, we prove that $|f| 1_{B_n} \to 0$. While it is clear that $|f| 1_{B_n} = 0$ on $\{|f| = 0\}$ for all n. If $|f(x)| > 0$, and $f_n(x) \to f(x)$, then there is N (depending on x in general) such that $|f_n(x) - f(x)| < \frac{1}{2} |f(x)|$ so that $f_n(x) f(x) > 0$ for all $n > N$, hence $x \notin B_n$ for $n > N$. Thus $1_{B_n}(x) = 0$ for all $n > N$, which yields that $|f| 1_{B_n}(x) = 0$ for all $n > N$. Since $f_n \to f$ almost surely, we thus can conclude that $|f| 1_{B_n} \to 0$ almost everywhere as $n \to \infty$. $|f| 1_{B_n}$ is controlled by the integrable function $|f|$, so that, by DCT we have $\int_{B_n} |f| d\mu = \int_{\Omega} |f| 1_{B_n} d\mu \to 0$.

Therefore, by Sandwich lemma for limits, it follows from (1) that $\lim_{n \to \infty} \int_{\mathbb{R}} |f_n - f| d\mu = 0$.

Q4. If $\rho \geq 0$ is continuous, then ρ is bounded on $[-n, n]$ for every n, thus there is M_n such that $|\rho(x)| \leq M_n$ for every $x \in [-n, n]$, so that

$$\mu([-n, n]) \leq 2nM_n < \infty$$

for every $n = 1, 2, \cdots$. Since $\bigcup_{n=1}^{\infty} [-n, n] = \mathbb{R}$, by definition, μ is σ-finite. μ is a finite measure if and only if $\mu(\mathbb{R}) = \int_{\mathbb{R}} \rho dm < \infty$, i.e. if and only if ρ is integrable.