Electromagnetism, Sheet 4: Time dependent electromagnetism and waves

1.- Starting from the Maxwell equations with sources[1], explain how to introduce the potentials, and what a gauge transformation is.

Assuming that a solution ζ of the following wave-equation-with-source

$$\frac{1}{c^2}\frac{\partial^2\zeta}{\partial t^2} - \nabla^2\zeta = F$$

exists for any F which you encounter, show that it is possible by means of a gauge transformation to impose the condition

$$\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \nabla\cdot\vec{A} = 0$$

on the potentials Φ and $\vec{A}$. (This is called *Lorenz gauge condition*). Show that the solution found in the lectures:

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c})}{|\vec{r}-\vec{r}\,'|}dV' \qquad (1)$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,',t-\frac{|\vec{r}-\vec{r}\,'|}{c})}{|\vec{r}-\vec{r}\,'|}dV' \qquad (2)$$

Satisfies this gauge.

2.- For the plane monochromatic wave (seen in lectures) with

$$\vec{E} = \vec{e}_1\cos\phi + \vec{e}_2\sin\phi,$$

where $\phi = \omega t - \vec{k}\cdot\vec{r}$, $\vec{B} = \vec{k}\wedge\vec{E}/\omega$, find relations between $\vec{e}_i, \vec{k}, \omega$ for the source-free Maxwell equations to hold.

What does it mean to say the wave is circularly polarised? In this case, calculate the energy density:

$$\mathcal{E} = \frac{1}{2}\epsilon_0|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2$$

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[1]

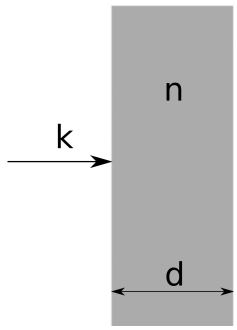
$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}; \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\wedge\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}; \qquad \nabla\wedge\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

and the Poynting vector:

$$\vec{P} = \frac{1}{\mu_0} \vec{E} \wedge \vec{B}$$

3.- A plane wave moving in a medium with refraction index one is incident from the left on a layered interface of refraction index n and thickness d, as shown in the figure



Assuming the electric field is polarized in the direction tangential to the interface, and the layered interface has magnetic permeability one, compute the transmission coefficient on the other side of the layer (ratio of transmitted Poynting's flux to incident Poynting's flux).

4.- Under Lorentz transformations for any constant $v < c$:

$$x' = \gamma(x - vt) \tag{3}$$

$$y' = y, \quad z' = z \tag{4}$$

$$t' = \gamma(t - \frac{v}{c^2}x) \tag{5}$$

The components of the electric and magnetic field transform as:

$$E'_x = E_x, \quad B'_x = B_x \tag{6}$$

$$E'_y = \gamma(E_y - vB_z), \quad B'_y = \gamma(B_y + \tfrac{v}{c^2}E_z) \tag{7}$$

$$E'_z = \gamma(E_z + vB_y), \quad B'_z = \gamma(B_z - \tfrac{v}{c^2}E_y) \tag{8}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Show that $\nabla' \cdot \vec{E}' = 0$ and $\nabla' \cdot \vec{B}' = 0$ follow from the source-free Maxwell equations for $\vec{E}$ and $\vec{B}$. (you can show the other equations also follow, but you are not asked to do so).