B6.3 Integer Programming, Problem Sheet 5 Solutions

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Problem 1. Consider the generalised assignment problem from Lecture 16,

$$
\begin{aligned}
\text{(GAP)} \quad \max_x \ & \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_{ij} \\
\text{s.t.} \ & \sum_{j=1}^{n} c_{ij} x_{ij} \le b_i, \quad (i = 1, \dots, m) \qquad (1) \\
& \sum_{i=1}^{m} x_{ij} = 1, \quad (j = 1, \dots, n) \qquad (2) \\
& x_{ij} \in \{0, 1\}, \quad (i = 1, \dots, m;\ j = 1, \dots, n).
\end{aligned}
$$

i) Dualise the knapsack constraints (1) and formulate the associated Lagrangian Dual problem. Prove that the optimal objective value w_{LD} of the Lagrangian dual equals the optimal value z_{LP} of the LP relaxation of (GAP). How would you compute a subgradient when applying the subgradient algorithm to this Lagrangian dual?

ii) Dualise the assignment constraints (2) and formulate the associated Lagrangian Dual problem. By giving an example, show that the optimal objective value w_{LD} of the Lagrangian dual is generally strictly smaller than the optimal value z_{LP} of the LP relaxation of (GAP). How would you compute a subgradient when applying the subgradient algorithm to this Lagrangian dual?

Problem 2.

i) Prove Proposition (Properties of the subdifferential) from Lecture 13, namely

   a) If f is differentiable at u, then $\partial f(u) = \{\nabla f(u)\}$ is a singleton containing only the gradient.
   b) $\partial f(u)$ is a convex set.
   c) $u^* \in \arg\min f(u)$ if and only if $\vec{0} \in \partial f(u^*)$.
   d) Let $z(u) := \max\{c^{\mathrm{T}} x + u^{\mathrm{T}}(d - Dx) : x \in \mathscr{X}\}$. Then

$$\partial z(u) = \operatorname{conv}\left(\left\{d - Dx^* : x^* \in \arg\max\{c^{\mathrm{T}} x + u^{\mathrm{T}}(d - Dx) : x \in \mathscr{X}\}\right\}\right),$$

   where arg max is the set of all maximisers.

ii) Set up a linear programming problem that can be used to check the optimality condition given in Part i.c).

Problem 3. In the lecture notes we proved Theorem (Characterisation of Lagrangian dual bound) only in the case where the feasible set $\mathscr{X}$ consists of finitely many points. One can prove that, more generally, the feasible set $\mathscr{X} = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0,\ x_i \in \mathbb{Z},\ (i \in \mathscr{I})\}$ (where $\mathscr{I}$ denotes the index set of variables that are integrality constrained) of any mixed integer programming problem has the property that there exist finitely many extreme points $x^{[1]}, \dots, x^{[k]}$ and finitely many extreme rays $v^{[1]}, \dots, v^{[\ell]}$ such that

$$\operatorname{conv}(\mathscr{X}) = \operatorname{conv}(\{x^{[j]} : j \in [1, k]\} + \operatorname{cone}(\{v^{[j]} : j \in [1, \ell]\}),$$

where $\mathrm{cone}(A) = \{\sum_{v\in V} \lambda_v v : V \subseteq A, |V| < \infty, \lambda_v > 0, v \in V\}$ denotes the set of linear combinations of finitely many elements of a set $A \subseteq \mathbb{R}^n$ with positive linear weights, and for any subsets $A, B \subseteq \mathbb{R}^n$, $A + B = \{v + w : v \in A, w \in B\}$ denotes the set of sums of elements of A with elements of B. Using this result or otherwise, prove Theorem (Characterisation of Lagrangian dual bound) in the general case (without assuming finiteness of $|\mathscr{X}|$, that is prove that

$$w_{LD} = \max\left\{c^{\mathrm{T}}x : D_1 x \le d_1, D_2 x = d_2, x \in \mathrm{conv}(\mathscr{X})\right\}.$$

Problem 4. Prove the following inequality, which was used in the proof of Lemma (Convergence of the subgradient algorithm) of Lecture 13:

$$\|u^{[k+1]} - u^*\|_2^2 \le \|u^{[k]} - \mu_k v^{[k]} - u^*\|_2^2$$

[*Hint: this is trivial in the case where all constraints are equality constraints, and* $u^{[k+1]} = u^{[k]} - \mu_k v^{[k]}$, *but more work needs to be done in the presence of inequality constraints. You may assume without loss of generality that all constraints are inequality constraints.*]

Problem 5. Give a proof of correctness for Algorithm (Minimal cover separation) from Lecture 16, that is, prove that its output is a minimal cover whose associated cover inequality is a cut for the point x^*.

Problem 6. Apply the cutting plane algorithm with Gomoroy cuts to the following IP,

$$\min_{x_1,x_2} \{x_1 + x_2 : 6x_1 + x_2 \le 4, 3x_1 \ge 1, x_1, x_2 \ge 0, x_1, x_2 \in \mathbb{Z}\}$$

Problem 7 (Wolsey Exercise 9.8.3). In each of the examples below, a set X and a point x^* are given. Find a valid inequality for X that cuts off x^* from the polyhedron used in the formulation of X.

i) $X = \{x \in \{0,1\}^5 : 9x_1 + 8x_2 + 6x_3 + 6x_4 + 5x_5 \le 14\}$, $x^* = (0, 5/8, 3/4, 3/4, 0)$.

ii) $X = \{x \in \{0,1\}^5 : 9x_1 + 8x_2 + 6x_3 + 6x_4 + 5x_5 \le 14\}$, $x^* = (1/4, 1/8, 3/4, 3/4, 0)$.

Problem 8. Consider problem

$$\text{(GAP)} \quad \max_x \sum_{i=1}^m \sum_{j=1}^n p_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{j=1}^n c_{ij} x_{ij} \le b_i, \quad (i = 1, \dots, m), \tag{3}$$

$$\sum_{i=1}^m x_{ij} = 1, \quad (j = 1, \dots, n), \tag{4}$$

$$x_{ij} \in \{0,1\}, \quad (i = 1, \dots, m; j = 1, \dots, n), \tag{5}$$

where $p_{ij}, c_{ij}, b_i \in \mathbb{Q}$ are fixed problem parameters for all i, j.

i) Suppose that the parameters of problem (GAP) are such that there exists an index i for which $c_{ij} > 0$ for all j and $b_i > 0$, and a set $C_i \subseteq \{1, \dots, n\}$ such that $\sum_{j\in C_i} c_{ij} > b_i$. Prove that all feasible solutions x of (GAP) must satisfy the inequality

$$\sum_{j\in C_i} x_{ij} \le |C_i| - 1. \tag{6}$$

ii) Denote the feasible set of problem (GAP) by $\mathscr{F}$, and assume that we are given an optimal solution x^* of the LP-relaxation of (GAP), and let (i^*, j^*) be the indices of its most fractional component x_{i^*,j^*}. Generalised Upper Bound branching (GUB) splits $\mathscr{F}$ into two disjoint sets according to the rule

$$\mathscr{F}_1 = \mathscr{F} \cap \{x : x_{ij^*} = 0, i = 1, \dots, p\}$$
$$\mathscr{F}_2 = \mathscr{F} \cap \{x : x_{ij^*} = 0, i = p+1, \dots, m\},$$

where $p := \min\{t : \sum_{i=1}^t x^*_{ij^*} \ge 1/2\}$. Discuss why this is not a good branching rule when $x^*_{mj^*} > 1/2$. Propose an algorithm to reorder the indices x_{ij^*} $(i = 1, \dots, m)$ so that GUB branching yields balanced sets $\mathscr{F}_1$ and $\mathscr{F}_2$, that is, sets of nearly equal cardinality.