Further Mathematical Biology: Problem Sheet 2

Michaelmas Term 2018

Morphogen Gradients and Positional Information.

Question 1.

The distribution of a morphogen M is described by the following equations:

$$\frac{\partial M}{\partial t} = D\frac{\partial^2 M}{\partial x^2} - \lambda M, \quad 0 < x < L$$

$$\text{with} \quad \frac{\partial M}{\partial x}(0,t) = 0, \;\; M(L,t) = M_L, \;\; M(x,0) = 0,$$

where D, λ, M_L and L are positive constants.

(a) Determine the steady state distribution of $M(x,t) \equiv M_s(x)$.

(b) Cells contained in the domain mature into cells of type I where $M_s \geq \theta$ and into cells of type II otherwise. Given that $0 < \theta < M_L$, determine the position $x_\theta \in [0, L)$ at which cells switch from type I to type II.

(c) Explain how x_θ changes as the domain size $L \to \infty$.

Question 2.

A population of stem cells occupy the tissue region $0 \leq x \leq L$. Their fate is determined by the steady state distributions of two morphogens, M and A. The concentrations of M and A are described by the following equations

$$0 = \frac{d^2 M}{dx^2}, \quad 0 = \frac{d^2 A}{dx^2} + \lambda M, \quad \text{for } 0 < x < L,$$

$$\text{with } M(0) = M_0, \; M(L) = 0 = A(0) = A(L).$$

(a) Determine the steady state distributions of M and A.

(b) The stem cells differentiate (i.e. mature) where $A > A^* > 0$. Show that a necessary condition for generating differentiated cells is that $L^2 > (9\sqrt{3}A^*)/\lambda M_0$.

(c) Explain why stems cells will always persist in the tissue.

Travelling waves.

Question 1.

Suppose fishing is regulated in a zone H km from a country's shore (taken to be a straight line), but outside this zone over-fishing is so excessive that the population is effectively zero. Assume that the fish reproduce logistically, disperse by diffusion and within the zone are harvested with an effort E.

(a) Justify the following model of the fish population $U(x,t)$:

$$\frac{\partial U}{\partial t} = rU\left(1 - \frac{U}{K}\right) - EU + D\frac{\partial^2 U}{\partial x^2}, \tag{1}$$

with boundary conditions

$$U = 0 \text{ on } x = H, \quad \frac{\partial U}{\partial x} = 0 \text{ on } x = 0,$$

where $r, K, E(< r)$ and D are positive constants.

(b) What are the spatially-uniform steady states of this system?

(c) Investigate the linear stability of the trivial steady state by writing

$$U = \epsilon U_1(x)e^{\lambda t} + O(\epsilon^2)$$

and determining the equation and boundary conditions satisfied by U_1.

(d) Assuming *real* growth-rates λ, determine a condition on parameters r, E, λ for a solution U_1 to exist, and assuming that this is satisfied, give the general solution for U_1. [Can *show* λ must be real, but tedious!]

(e) If the fish stock is not to collapse (*i.e.* the trivial solution is unstable), show that the fishing zone H must satisfy

$$H > \frac{\pi}{2} / \left(\frac{(r-E)}{D}\right)^{1/2}.$$

[**Hint:** you need to determine conditions for which the trivial solution $U = 0$ is *unstable.*]

(f) Discuss briefly the ecological implications of this result.

Question 2.

Consider a population of cells $u = u(x,t)$ that undergoes logistic growth but whose diffusion depends linearly on their their density so that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u(1-u).$$

Seek travelling wave solutions of the form $u(x,t) = U(z)$ where $z = x - ct$ and $c > 0$. Determine the equations satisfied by $U(z)$ and $V(z) = U'(z)$. By looking for a solution of the form $V = c(U-1)$, show that there is an exact solution for one value of the wave speed, c. You should state this value of c.

Question 3.

A non-dimensionalised version of the red squirrel-grey squirrel competition model takes the form

$$\begin{aligned}\frac{\partial G}{\partial t} &= \frac{\partial^2 G}{\partial x^2} + G(1 - G - \gamma_1 R),\\ \frac{\partial R}{\partial t} &= K\frac{\partial^2 R}{\partial x^2} + \alpha R(1 - R - \gamma_2 G),\end{aligned}$$

where $G(x,t)$ and $R(x,t)$ are the densities of grey and red squirrels, respectively, at time t and position x and K, α, γ_1 and γ_2 are positive constants with $\gamma_1 < 1$ and $\gamma_2 > 1$.

We seek travelling wave solutions of the form $G = G(z)$, $R = R(z)$ where $z = x - Vt$ and V is a positive constant, with $G(-\infty) = 1$, $R(-\infty) = 0$, $G(\infty) = 0$ and $R(\infty) = 1$.

(a) Write down the system in travelling wave coordinates.

(b) In the special case where $K = 1$, $\alpha = 1$, $\gamma_1 + \gamma_2 = 2$, show that $S = G + R$ satisfies the equation

$$\frac{\mathrm{d}^2 S}{\mathrm{d}z^2} + V\frac{\mathrm{d}S}{\mathrm{d}z} + S(1 - S) = 0,$$

and hence, by considering boundary conditions, that $S \equiv 1$ for all z is a solution.

(c) Deduce that, for this special case,

$$\frac{\mathrm{d}^2 G}{\mathrm{d}z^2} + V\frac{\mathrm{d}G}{\mathrm{d}z} + (1 - \gamma_1)G(1 - G) = 0.$$

(d) Show that, for this equation, travelling waves are possible if $V \geq 2(1 - \gamma_1)^{1/2}$ and sketch the wave.

Question 4.

A rabies model which includes logistic growth for the susceptibles, S, and diffusive dispersal for the infectives, I, is

$$\begin{aligned}\frac{\partial I}{\partial t} &= D\frac{\partial^2 I}{\partial x^2} + rIS - aI,\\ \frac{\partial S}{\partial t} &= -rIS + BS\left(1 - \frac{S}{S_0}\right),\end{aligned}$$

where r, a, B, D and s_0 are positive parameters.

(a) Non-dimensionalise the system to give

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial y^2} + uv - \lambda u,\\ \frac{\partial v}{\partial \tau} &= -uv + bv(1 - v),\end{aligned}$$

where u relates to I and v to S.

(b) Look for travelling wave solutions with $u > 0$ and $v > 0$ and hence show, by linearising far ahead of a wavefront, where $v \to 1$ and $u \to 0$, *i.e.* far ahead where the population is still fully susceptible and the infection has not yet arrived, that a wave may exist if $\lambda < 1$ and, if so, the wave speed is $c \geq 2\sqrt{(1 - \lambda)}$.