Note: These problems are for practice and revision purposes. This sheet is not to be turned in.

1. Consider the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

subject to $u(x,0) = u(x,a) = 0$. Solve via separation of variables.

Solution: *If we seek a solution of the form*

$$u(x,y) = X(x)Y(y),$$

then the equation becomes

$$X''(x)Y(y) - X(x)Y''(y) = 0 \quad \rightarrow \quad \frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)}.$$

Since the LHS of this equation depends only on x and the RHS depends only on y, both sides must be equal to a constant, λ say. Then the PDE transforms to the following pair of second order ODEs:

$$X'' - \lambda X = 0 = Y'' - \lambda Y.$$

What properties does the constant λ have? If $\lambda = 0$ then we have the trivial solution. If $\lambda > 0$ then the boundary conditions will give the trivial solution. So, we suppose that $\lambda = -k^2 < 0$ and deal with the equation (which has homogeneous boundary conditions), i.e. solve

$$Y'' - \lambda Y = 0 = Y'' + k^2 Y \ \text{ subject to } \ Y(0) = Y(a) = 0.$$

This gives the general solution $Y(y) = A\cos(ky) + B\sin(ky)$. Now, boundary conditions are $Y(0) = Y(a) = 0$ so then

$$A\cos(k0) + B\sin(k0) = 0$$

$$A\cos(ka) + B\sin(ka) = 0.$$

For non-trivial solutions we require that the determinant $(AB\cos k0 \sin ka - AB\sin k0 \cos ka) = 0$, which means $\sin ka = 0$, so $ka = n\pi$ for $n = 1, 2, \ldots$ or $k = n\pi/a$ for $n = 1, 2, \ldots$. To satisfy the boundary conditions, we require $A = 0$ and we take $B = 1$, without loss of generality. Hence we have an infinite set of solutions for $Y(y)$, i.e.

$$\left\{Y_n(y) = \sin\left(\frac{\pi n y}{a}\right), \ \text{for } n = 1, 2, \ldots\right\},$$

each of which satisfies the boundary conditions. It remains to find a set $X_n(x)$ so that we can find an expression for the solution of the form $u_n(x,y) = X_n(x)Y_n(y)$. We solve $X'' - \frac{\pi n}{a}X = 0$ to give $X_n(x) = C_n\cos(\frac{\pi n x}{a}) + D_n\sin(\frac{\pi n x}{a})$. Now we can write the general solution $u(x,y)$ in the form of an infinite sum:

$$u(x,y) = \sum_{n=1}^{\infty}\left\{C_n\cos\left(\frac{\pi n x}{a}\right) + D_n\sin\left(\frac{\pi n x}{a}\right)\right\}.$$

The arbitrary constants C_n and D_n may be found by Fourier analysis if u and $\frac{\partial u}{\partial x}$ are given on (say) $x = 0$.

2. Solve Laplace's Equation, with boundary conditions as shown:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } x \in (0,\pi),\ y \in (0,\pi),$$
$$u(0,y) = 0, u(\pi,y) = 0,$$
$$u(x,0) = 0, u(x,\pi) = \sin^3 x.$$

Solution: *Separation of variables leads to the solution:*

$$u(x,y) = \frac{3}{4}\sin(x)\frac{\sinh(y)}{\sinh(\pi)} - \frac{1}{4}\sin(3x)\frac{\sinh(3y)}{\sinh(3\pi)}.$$

3. Solve the Heat Equation, with boundary conditions as shown:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } x \in (0,\pi),\ t > 0,$$
$$u(x,0) = \sin^3 x, \quad u(0,t) = u(\pi,t) = 0 \ \ \forall\, t > 0,$$
$$u \to 0 \quad \text{as } t \to \infty.$$

Solution: *Separation of variables leads to the solution:*

$$u(x,t) = \frac{3}{4}\sin(x)e^{-t} - \frac{1}{4}\sin(3x)e^{-9t}.$$

4. Recall: the Fourier transform of u is defined by

$$\hat{u}(t,k) = \int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \tag{1}$$

and u may be recovered from $\hat{u}$ by using the *inversion formula*

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(t,k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x. \tag{2}$$

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{3}$$

subject to $u \to 0$ as $x \to \pm\infty$ and $u = u_0(x)$ when $t = 0$. Solve by taking a Fourier transform in x.

Solution: *The heat equation is transformed to*

$$\frac{\partial \hat{u}}{\partial t} = -k^2\hat{u} \qquad \Rightarrow \qquad \hat{u} = \hat{u}_0\mathrm{e}^{-k^2 t}, \tag{4}$$

where $\hat{u}_0$ *is the Fourier transform of* u_0*. Then the* convolution theorem *gives*

$$u(x,t) = u_0(x) * f(x,t) = \int_{-\infty}^{\infty} u_0(\xi) f(x-\xi,t)\,\mathrm{d}\xi, \tag{5}$$

where

$$\hat{f}(t,k) = \mathrm{e}^{-k^2 t}. \tag{6}$$

It is straightforward to invert this transform and thus find

$$f(x,t) = \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right). \tag{7}$$

(This is the Green's function *for the heat equation.)*

5. Consider the boundary value problem

$$Ly(x) = f(x) \quad \text{on } 0 < x < 1, \qquad y'(0) + y(0) = \alpha, \quad y(1) = \beta, \tag{8}$$

with L is the differential operator given by

$$Ly \equiv y''(x) + 4y(x), \tag{9}$$

Derive a problem for the Green's function $g(x, \xi)$ in terms of the delta function $\delta(x)$ and give the form of the solution in terms of g.

Solution: *Multiply the equation $Ly = f$ by $g(x, \xi)$ on both sides and integrate over the domain. The left hand side becomes, after integrating by parts twice (note that L is a self-adjoint operator):*

$$\int_0^1 Ly(x)g(x,\xi)\,dx = (y'g - yg')\big|_0^1 + \int_0^1 y(x)Lg(x,\xi)\,dx.$$

If g satisfies

$$Lg(x,\xi) = \delta(x - \xi),$$

then by property of the delta function,

$$\int_0^1 y(x)Lg(x,\xi)\,dx = y(\xi).$$

Further, we choose g to satisfy homogeneous boundary conditions:

$$g_x(0,\xi) + g(0,\xi) = 0, \quad g(1,\xi) = 0,$$

Combining the conditions on g and the non-homogeneous boundary conditions on y, we get the solution

$$y(\xi) = \int_0^1 g(x,\xi)f(x)\,dx + \alpha g(0,\xi) + \beta g_x(1,\xi)$$