1. Use Charpit's method to solve the partial differential equation

$$u_x^2 + yu_y = u$$

subject to the initial data $u(x, 1) = 1 + x^2/4$ for $-\infty < x < \infty$. Express the solution in explicit form and determine where it is uniquely defined by the data.

2. Find all solutions of the partial differential equation

$$u_x u_y = 1$$

that satisfy the initial data $u = 0$ when $x + y = 1$, and state where each solution exists.

Given general initial data $u = u_0(s)$, $x = x_0(s)$, $y = y_0(s)$, show that a necessary condition for a real solution to exist is $(u_0')^2 \geq 4x_0' y_0'$.

3. Derive Charpit's equations for the PDE

$$F(x, y, u_x, u_y) = 0.$$

Show that these equations describe straight rays in the case

$$F = u_x^2 + u_y^2 - 1.$$

Suppose that $u = 0$ on the boundary of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Find and draw the envelope of the rays that propagate into the ellipse, and determine the ridge of discontinuity. *Note: consider non-local intersections of rays that will not be found in the envelope!*

4. Verify that the solution of

$$\dot{x} = F_p, \quad \dot{y} = F_q, \quad \dot{u} = pF_p + qF_q, \quad \dot{p} = \dot{q} = 0,$$

with suitable initial data on $t = 0$, satisfies $F(p, q) = 0$, and that

$$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}.$$

5. A plane wave of light, moving in the x-direction, reflects off the right half of the unit circle $\Gamma : x_0(s) = \cos(s), y_0(s) = \sin(s)$. Supposing that the phase of the reflected wave

$$\phi_R = Ae^{iu(x,y)}$$

satisfies the Eikonal equation $|\nabla u|^2 = 1$ and boundary condition $u = x$ on Γ, find and plot the caustic, i.e. the envelope of the reflected rays. *[You may use a computer algebra program to compute and plot the envelope.]*

6. (Bookwork) Show that, if $\mathbf{u}(x, y)$ is a continuous solution of the quasi-linear system

$$\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y = \mathbf{c}$$

and $\mathbf{u}_x$, $\mathbf{u}_y$ have jumps across a curve given by $y = y(x)$, then

$$\det\left(\mathbf{B} - \frac{dy}{dx}\mathbf{A}\right) = 0.$$

7. Show that the system

$$\begin{aligned} u_x + u_y + v_y &= 0, \\ v_x + u_y + 2v_y + w_y &= 0, \\ w_x - u_y + 2v_y &= 0, \end{aligned}$$

is hyperbolic and evaluate its Riemann invariants. Hence find the general solution.

8. If u and v satisfy the PDEs

$$\begin{aligned} (\cos v)u_x + (\sin v)u_y - v_y &= 0, \\ (\cos v)v_x - u_y + (\sin v)v_y &= 0, \end{aligned}$$

find the characteristic directions and Riemann invariants.