1. In two-dimensional steady homentropic gas flow, the velocity (u, v) and density ρ satisfy the equations

$$\begin{aligned}(\rho u)_x + (\rho v)_y &= 0,\\ \rho\left(uu_x + vu_y\right) + c(\rho)^2\rho_x &= 0,\\ \rho\left(uv_x + vv_y\right) + c(\rho)^2\rho_y &= 0,\end{aligned}$$

where $c(\rho)$ is the *speed of sound* in the gas.

Show that the system is hyperbolic if the flow is *supersonic*, *i.e.* if $u^2 + v^2 > c^2$.

Also prove *Bernoulli's theorem*, that the Riemann invariant

$$R = \frac{1}{2}\left(u^2 + v^2\right) + \int \frac{c(\rho)^2\,\mathrm{d}\rho}{\rho}$$

is conserved along the characteristics $\mathrm{d}y/\mathrm{d}x = v/u$.

2. Reduce the following PDE to canonical form and then determine its general solution:

$$y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{1}{xy}\left(y^3u_x + x^3u_y\right).$$

3. Solve the PDE

$$u_{xx} + u_{xy} - 2u_{yy} + 1 = 0 \ \text{ in } 0 \le x \le 1,\ y > 0, \text{ with } u = u_y = x \text{ on } y = 0.$$

4. Reduce the following PDE to canonical form:

$$u_{xx} + u_{xy} + u_{yy} = 0.$$

5. Consider the following system of PDEs for $u(x, y)$, $v(x, y)$

$$\begin{aligned}&2x^2yu_x + 5xy^2u_y + 2x^2y^2v_y + 5xyu + x = 0,\\ &yu_y - x^2v_x + u - 2xv = 0.\end{aligned}$$

- In the region $x > 0, y > 0$, determine whether the system is hyperbolic, elliptic, or parabolic.
- Show that the system is equivalent to a second order semilinear PDE.

6. Determine the characteristic coordinates $\xi(x, y)$ and $\eta(x, y)$ for the system

$$xu_{xx} + u_{yy} = 0$$

for the regions (i) $x < 0$ and (ii) $x > 0$.

7. (i) Show that, if

$$\phi_{xx} - \phi_{tt} = f(x, t),$$

with

$$\phi = \phi_t = 0 \quad \text{at} \quad t = 0,$$

then

$$\phi(\xi, \tau) = \frac{1}{2} \iint_{\Delta} f(x, t) \, \mathrm{d}x\mathrm{d}t,$$

where Δ is a triangle which you should specify.

(ii) Solve

$$\phi_{xx} - \phi_{tt} = -1 \quad \text{in} \quad x > 0, \ t > 0,$$

with

$$\phi = \phi_t = 0 \quad \text{on} \quad t = 0, \ x > 0,$$

and

$$\phi = 0 \quad \text{on} \quad x = 0, \ t > 0.$$

Show that there is a discontinuity of magnitude 1 in $\partial^2\phi/\partial x^2$ across $x = t$.

8. Find the Riemann function for the PDE

$$u_{xy} + xyu_x = f(x, y)$$

and use it show that the solution to the problem

$$\begin{aligned} u_{xy} + xyu_x &= 0 \quad \text{in} \quad x + y > 0, \\ u = x, \quad u_y &= 0 \quad \text{on} \quad x + y = 0. \end{aligned}$$

is

$$u(\xi, \eta) = -\eta + \int_{-\eta}^{\xi} e^{x(x^2 - \eta^2)/2} dx.$$