1. Suppose that $\phi(x,t;\tau)$ is the solution of

$$\phi_{tt} - \phi_{xx} = 0, \quad \text{with} \quad \phi(x,\tau;\tau) = 0, \quad \phi_t(x,\tau;\tau) = g(x,\tau).$$

Construct a solution for ϕ involving an integral of g and show that

$$u(x,t) = \int_0^t \phi(x,t;\tau)\,\mathrm{d}\tau$$

satisfies the inhomogeneous problem

$$u_{tt} - u_{xx} = g(x,t) \quad \text{with} \quad u(x,0) = u_t(x,0) = 0.$$

2. Consider the problem

$$\nabla^2\phi = 0 \quad \text{for } 1 \le r \le 2, \quad \text{with} \quad \alpha\phi + \frac{\partial\phi}{\partial r} = \begin{cases} k\cos\theta & \text{on} \quad r = 1, \\ 0 & \text{on} \quad r = 2. \end{cases}$$

for constant α.

(i) If $\alpha = 0$, show that the solvability condition for existence of a solution is satisfied. *(Use an extension of Green's theorem for a non-simply-connected domain.)*

(ii) Show that the homogeneous problem (i.e. $k = 0$) has a solution $u = u(r)$ if

$$\alpha = 0 \quad \text{or} \quad \alpha = \frac{1}{2\log 2}.$$

(iii) By seeking a solution of the homogeneous problem of the form $u = f(r)g(\theta)$, show that that there are countably infinitely many such solutions (i.e. countably infinitely many α for which such solutions exist).

(iv) What can you say about existence and uniqueness of the inhomogeneous problem as a function of α?

3. Construct the Green's function for Laplace's equation in the domain $x > 0, y > 0$ with Neumann boundary data. Hence, assuming suitable behaviour at infinity, give a solution to the problem

$$\begin{aligned} \nabla^2 u &= f && \text{in } x > 0, y > 0, \\ \frac{\partial u}{\partial x} &= g(y) && \text{on } x = 0, \\ \frac{\partial u}{\partial y} &= h(x) && \text{on } y = 0. \end{aligned}$$

4. Let w be the difference between two solutions of

$$\begin{aligned} &\frac{\partial u}{\partial t} = \nabla^2 u + au + f(\mathbf{x},t) \quad \mathbf{x} \in \Omega, t > 0 \\ &\nabla u \cdot \mathbf{n} + \alpha u = g(\mathbf{x},t) \quad \mathbf{x} \in \partial\Omega \\ &u = h(\mathbf{x}) \quad \text{at } t = 0 \end{aligned}$$

where a and α are constants with $\alpha > 0$, and $\mathbf{n}$ is the unit normal on $\partial\Omega$. Derive the relation

$$\frac{d}{dt}\int_{\Omega} w^2 d\mathbf{x} + 2\int_{\Omega} \left(|\nabla w|^2 - aw^2\right) d\mathbf{x} + 2\alpha \int_{\partial\Omega} w^2 dS = 0$$

Deduce that

$$\left(\frac{d}{dt} + \text{constant}\right)\int_{\Omega} w^2 d\mathbf{x} \le 0$$

and thus that $w \equiv 0$.

5. Consider the equation

$$u_t = u_{xx}, \quad x > 0, \ t > 0$$

with

$$u(x,0) = 0, \quad u(0,t) = f(t)$$

(i) Explain why this admits a *similarity solution* when f is constant. Thus obtain the solution $u = u_0(x,t)$ corresponding to $f \equiv 1$.
You may find useful the formula

$$\int_0^{\infty} e^{-s^2/4}\, ds = \sqrt{\pi}.$$

(ii) Use the Green's function approach to show that the solution for arbitrary $f(t)$ may be written in the form

$$u = \int_0^t f(t-s)\, \frac{\partial u_0}{\partial t}(x,s)\, \mathrm{d}s.$$

6. Find a similarity solution of the equation, for constant $\alpha \in (0,1)$,

$$u_t = x^{\alpha} u_{xx}, \quad \text{for } x, t > 0,$$

which also satisfies the boundary conditions

$$u(0,t) = 0, \ \ u(x,0) = T_0 > 0, \ \text{ and } \ u \to T_0 > 0 \ \text{ as } x \to \infty.$$

7. Find a non-trivial similarity solution of the equation

$$u_t = (uu_x)_x \ \text{ in } \ 0 < x < t^{1/3}, \ t > 0,$$

where

$$u_x(0,t) = 0 = u(t^{1/3}, t) \ \text{ for } \ t > 0.$$

Show that

$$\int_0^{t^{1/3}} u(x,t) dx = \ \text{constant for } t > 0.$$