1. (a) [9 marks] State the definitions of *test function* $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and *distribution* $u \in \mathcal{D}'(\mathbb{R}^n)$.
We now assume that $u \in \mathcal{D}'(\mathbb{R})$. Explain how one should define its *distributional derivative* u' and show that it reduces to the usual concept for C^1 functions.
Using that there exists a nonnegative test function $\theta \in \mathcal{D}(\mathbb{R})$ with support in $[-1, 1]$ and integral $\int_{\mathbb{R}} \theta \mathrm{d}x = 1$ show that if $u \in \mathcal{D}'(\mathbb{R})$ and $u' = 0$, then $u = c$ for some constant $c \in \mathbb{R}$.
Let $a\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function and $u \in \mathcal{D}'(\mathbb{R})$ a distribution. Define the product au and state and prove the *Leibniz formula*.

(b) [4 marks] For $f,\ g \in \mathrm{C}^1(\mathbb{R})$ we define

$$u(x) = \begin{cases} f(x) & \text{if } x < 1 \\ g(x) & \text{if } x \geqslant 1. \end{cases}$$

Explain why $u \in \mathcal{D}'(\mathbb{R})$ and calculate its distributional derivative u'.

(c) [12 marks] (i) Solve the equation

$$u' + 2xu = \delta_0$$

for $u \in \mathcal{D}'(\mathbb{R})$, where δ_0 is Dirac's delta function at 0.

(ii) Let $f \in \mathcal{D}(\mathbb{R})$. Solve the equation

$$v' + 2xv = f$$

for $v \in \mathcal{D}'(\mathbb{R})$.

(iii) What is the general solution to

$$w' - 2xw = 0$$

in the space of *tempered distributions* $w \in \mathcal{S}'(\mathbb{R})$?

2. (a) [9 marks] For each $r > 0$ we define the r-dilation of a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by the rule

$$(d_r\varphi)(x) = \varphi(rx), \quad x \in \mathbb{R}^n.$$

(i) Extend the r-dilation to distributions $u \in \mathcal{D}'(\mathbb{R}^n)$.

(ii) Let $\alpha \in (-n, \infty)$ and $u_\alpha(x) = |x|^\alpha$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that $u_\alpha \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ and conclude that $u_\alpha \in \mathcal{D}'(\mathbb{R}^n)$. Prove that $d_r u_\alpha = r^\alpha u_\alpha$ for all $r > 0$. We express this by saying that u_α is *homogeneous of degree* α.

(iii) Show that the Dirac delta function δ_0 concentrated at the origin $0 \in \mathbb{R}^n$ is homogeneous of degree $-n$.

(b) [8 marks] Define for each $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\left\langle \mathrm{pv}\Big(\frac{1}{x}\Big), \varphi \right\rangle = \lim_{a \to 0^+} \left(\int_{-\infty}^{-a} + \int_a^\infty \right) \frac{\varphi(x)}{x}\, \mathrm{d}x.$$

Show that hereby $\mathrm{pv}\big(\frac{1}{x}\big) \in \mathcal{D}'(\mathbb{R})$ and that it is homogeneous of order -1 (as defined in (a)). Check that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log|x| = \mathrm{pv}\Big(\frac{1}{x}\Big).$$

(c) [8 marks] Show that $u = \mathrm{pv}\big(\frac{1}{x}\big)$ solves the equation

$$xu = 1 \tag{1}$$

in the sense of $\mathcal{D}'(\mathbb{R})$. What is the general solution $u \in \mathcal{D}'(\mathbb{R})$ to (1)?

3. (a) [12 marks] (i) What does it mean to say that φ is a *Schwartz test function on* $\mathbb{R}$ and that a sequence (φ_j) converges to φ in $\mathcal{S}(\mathbb{R})$? State your definitions in terms of the norms

$$\overline{S}_{k,l}(\varphi) := \sup\left\{ |x^r \varphi^{(s)}(x)| : r \in \{0,1,\dots,k\},\, s \in \{0,1,\dots,l\},\, x \in \mathbb{R} \right\}$$

where $k,\, l \in \mathbb{N}_0$. What does it mean to say that u is a *tempered distribution on* $\mathbb{R}$? Define the *Fourier transform* $\mathcal{F}(f) = \hat{f}$ of $f \in \mathrm{L}^1(\mathbb{R})$.

Show that $\mathcal{S}(\mathbb{R}) \subset \mathrm{L}^1(\mathbb{R})$ and state the *Fourier Inversion Formula on* $\mathcal{S}(\mathbb{R})$.

State the boundedness property of the Fourier transform $\varphi \mapsto \hat{\varphi}$ on the space $\mathcal{S}(\mathbb{R})$ of Schwartz test functions on $\mathbb{R}$ in terms of the family of norms $\overline{S}_{k,l}$, $k,\, l \in \mathbb{N}_0$.

(ii) Define the Fourier transform of a tempered distribution on $\mathbb{R}$, and deduce the Fourier Inversion Formula on $\mathcal{S}'(\mathbb{R})$ from its version on $\mathcal{S}(\mathbb{R})$.

Let u_k and u be tempered distributions on $\mathbb{R}$. What does it mean to say that $\sum_{k=1}^{\infty} u_k = u$ in $\mathcal{S}'(\mathbb{R})$?

Show, for instance by use of the boundedness property of the Fourier transform on $\mathcal{S}(\mathbb{R})$, that there exist constants $c > 0$, $m,\, n \in \mathbb{N}_0$ so that

$$\left|\hat{\varphi}(k)\right| \leqslant c\overline{S}_{m,n}(\varphi)\tfrac{1}{k^2}$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R})$ and $k \in \mathbb{Z} \setminus \{0\}$. Show that the series

$$\sum_{k=1}^{\infty} \mathrm{e}^{\mathrm{i}kx} \quad \text{and} \quad \sum_{k=1}^{\infty} \delta_k \tag{2}$$

both converge in $\mathcal{S}'(\mathbb{R})$. Here δ_k is Dirac's delta-function concentrated at k.

(iii) For $\varepsilon > 0$ define the tempered distribution T_ε by the rule

$$\langle T_\varepsilon, \varphi \rangle = \int_{-\infty}^{\infty} \left(\tfrac{1}{x+\varepsilon \mathrm{i}} - \tfrac{1}{x-\varepsilon \mathrm{i}} \right) \varphi(x)\, \mathrm{d}x, \quad \varphi \in \mathcal{S}(\mathbb{R})$$

Show, for instance by first considering $\mathrm{Log}(x+\varepsilon\mathrm{i}) - \mathrm{Log}(x-\varepsilon\mathrm{i})$, where Log denotes the principal logarithm, that $T_\varepsilon \to -2\pi\mathrm{i}\delta_0$ in $\mathrm{S}'(\mathbb{R})$ as $\varepsilon \to 0^+$. [*Results about differentiation of distributions and limit theorems from Integration Theory may be used without careful justification.*]

(b) [6 marks] Find the Fourier transform of δ_k. What is the relation between the sums of the two series in (2) above? Show that

$$\frac{1}{\mathrm{e}^{\varepsilon - \mathrm{i}x} - 1} \to \sum_{k=1}^{\infty} \mathrm{e}^{\mathrm{i}kx} \quad \text{and} \quad \frac{1}{1 - \mathrm{e}^{-\varepsilon - \mathrm{i}x}} \to \sum_{k=0}^{\infty} \mathrm{e}^{-\mathrm{i}kx} \quad \text{in} \quad \mathcal{S}'(\mathbb{R})$$

as $\varepsilon \to 0^+$.

(c) [7 marks] We now consider the sum

$$\sum_{k \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}kx} := \sum_{k=1}^{\infty} \mathrm{e}^{\mathrm{i}kx} + \sum_{k=0}^{\infty} \mathrm{e}^{-\mathrm{i}kx}$$

in the sense of tempered distributions on $\mathbb{R}$. For a test function $\varphi \in \mathcal{D}(\mathbb{R})$ supported in the interval $(-2\pi, 2\pi)$ show that

$$\left\langle \frac{1}{1 - \mathrm{e}^{-\varepsilon - \mathrm{i}x}} - \frac{1}{1 - \mathrm{e}^{\varepsilon - \mathrm{i}x}}, \varphi \right\rangle \to 2\pi\varphi(0)$$

as $\varepsilon \to 0^+$. [*Note that* $\frac{1}{1-\mathrm{e}^z} = -\frac{1}{z} + h(z)$ *for all* $0 < |z| < 2\pi$, *where* $h(z)$ *is a holomorphic function in* $|z| < 2\pi$.]

Deduce that

$$\sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}kx} = 2\pi \sum_{k\in\mathbb{Z}} \delta_{2\pi k} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}).$$

[*You may use localization results for distributions without careful justification.*]