1 (a)

$\varphi \in \mathcal{D}(\mathbb{R}^n)$ if $\varphi \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(\varphi) = \overline{\{x : \varphi(x) \neq 0\}}$ is cpt.

$u \in \mathcal{D}'(\mathbb{R}^n)$ if $u : \mathcal{D}(\mathbb{R}^n) \to \mathbb{R}$ is linear and $\forall$ cpt $K \subset \mathbb{R}^n$ $\exists\, m \in \mathbb{N}_0$, $c > 0$ s.t.

$$|\langle u, \varphi\rangle| \leq c \sup_{\substack{|\alpha| \leq m \\ x \in K}} |\varphi^{(\alpha)}(x)| \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n) \text{ with } \operatorname{supp}(\varphi) \subseteq K.$$

For C^1 function $u : \mathbb{R} \to \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R})$ we have by partial integration

$$\langle u', \varphi\rangle = \int_{-\infty}^{\infty} u'(x)\varphi(x)\,dx = \Big[u(x)\varphi(x)\Big]_{x \to -\infty}^{x \to \infty} - \int_{-\infty}^{\infty} u\varphi'\,dx$$

$$= \langle u, -\varphi'\rangle.$$

For $u \in \mathcal{D}'(\mathbb{R})$ we define $\langle u', \varphi\rangle := \langle u, -\varphi'\rangle$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Then clearly $u' : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ linear and if $K \subset \mathbb{R}$ cpt then $\exists\, m \in \mathbb{N}_0, c > 0$ s.t. $|\langle u, \varphi\rangle| \leq c \sup_{\substack{|\alpha| \leq m \\ K}} |\varphi^{(\alpha)}(x)| \quad \forall \varphi \in \mathcal{D}_K(\mathbb{R})$

$\mathcal{D}_K(\mathbb{R}) = \mathcal{D}(\mathbb{R}) \cap \{\varphi : \operatorname{supp}\varphi \subseteq K\}$

Now $|\langle u', \varphi\rangle| = |\langle u, -\varphi'\rangle| \leq c \sup_{|\alpha| \leq m,\, K} |\varphi^{(\alpha+1)}(x)|$ for $\varphi \in \mathcal{D}(K)$. $\therefore$ $u' \in \mathcal{D}'(\mathbb{R})$.

When $u \in C^1(\mathbb{R})$ we clearly have that the distributional derivative agrees with the usual derivative:

$$\langle Du, \varphi \rangle = \langle u, -\varphi' \rangle = \langle u', \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R})$$

and so $Du = u'$ in $\mathcal{D}'(\mathbb{R})$ (and they also agree as C^1 functions by Fundamental Lemma of CoV).

Let $\varphi \in \mathcal{D}(\mathbb{R})$ and put $I = \int_{\mathbb{R}} \varphi \, dx$.

If $\psi(x) = \int_{-\infty}^{x} \left(\varphi(t) - I\theta(t) \right) dt$, $x \in \mathbb{R}$, then $\psi \in C^1(\mathbb{R})$ by FTC with $\psi' = \varphi - I\theta$, so $\psi \in C^\infty(\mathbb{R})$. If $\mathrm{supp}(\varphi), \mathrm{supp}(\theta) \subset [-r, r]$, then for $x \leq -r$, $\psi(x) = 0$, while for $x \geq r$, $\psi(x) = \int_{-\infty}^{\infty} (\varphi - I\theta)\, dt = 0$. Thus $\mathrm{supp}(\psi) \subset [-r, r]$ and therefore $\psi \in \mathcal{D}(\mathbb{R})$.

Now $\varphi = \psi' + I\theta$, so

$$\langle u, \varphi \rangle = \langle u, \psi' \rangle + I \langle u, \theta \rangle = c \int_{\mathbb{R}} \varphi \, dx$$

with $c := \langle u, \theta \rangle$.

Let $a \in C^\infty(\mathbb{R})$, $u \in \mathcal{D}'(\mathbb{R})$. Then

$$\langle au, \varphi \rangle := \langle u, a\varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Well-defined extension of usual product by adjoint identity scheme, and $au \in \mathcal{D}'(\mathbb{R})$ is check by Leibniz rule in $\mathcal{D}(\mathbb{R})$:

If $\varphi \in \mathcal{D}(\mathbb{R})$, $\mathrm{supp}(\varphi) \subset K$, so

$$|\langle u, \varphi \rangle| \leqq c \sup_{\substack{0 \leqq j \leqq m \\ K}} |\varphi^{(j)}(x)|$$

we get since $a\varphi \in \mathcal{D}(K)$,

$$|\langle au, \varphi \rangle| = |\langle u, a\varphi \rangle| \leqq c \sup_{\substack{0 \leqq j \leqq k \\ K}} |(a\varphi)^{(j)}(x)|$$

and

$$(a\varphi)^{(j)}(x) = \sum_{s=0}^{j} \binom{j}{s} a^{(s)} \varphi^{(j-s)}$$

$$\leqq 2^k \max_{\substack{0 \leqq j \leqq k \\ x \in K}} |a^{(j)}(x)| \cdot \sup_{\substack{0 \leqq j \leqq k \\ x \in K}} |\varphi^{(j)}(x)|.$$

Leibniz Rule in $\mathcal{D}'(\mathbb{R})$: $(au)' = a'u + au'$

$\boxed{\text{Pf}}$ For $\varphi \in \mathcal{D}(\mathbb{R})$

$$\langle (au)', \varphi \rangle = \langle au, -\varphi' \rangle = \langle u, -a\varphi' \rangle$$

$$\langle a'u + au', \varphi \rangle = \langle u, a'\varphi \rangle + \langle u', a\varphi \rangle =$$

$$\langle u, a'\varphi \rangle - \langle u, (\varphi a)' \rangle = \langle u, -a\varphi' \rangle. \square$$

(b) $f, g \in C^1(\mathbb{R})$ and

$$u(x) = \begin{cases} f(x) & x < 1 \\ g(x) & x \geqq 1. \end{cases}$$

Then $u \in L^1_{loc}(\mathbb{R})$ so in particular $u \in \mathcal{D}'(\mathbb{R})$ by definition $\langle u, \varphi \rangle := \int_{\mathbb{R}} u\varphi \, dx$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Now $\langle u', \varphi \rangle = \langle u, -\varphi^* \rangle =$

$$-\int_{-\infty}^{1} f(x)\varphi'(x)\,dx - \int_{1}^{\infty} g(x)\varphi'(x)\,dx \overset{\text{parts}}{=}$$

$$-\Big[f(x)\varphi(x)\Big]_{x\to-\infty}^{x=1} + \int_{-\infty}^{1} f'(x)\varphi(x)\,dx - \Big[g(x)\varphi(x)\Big]_{x=1}^{x\to\infty} + \int_{1}^{\infty} g'(x)\varphi(x)\,dx$$

$$= -f(1)\varphi(1) + \int_{-\infty}^{1} f'\varphi\,dx + g(1)\varphi(1) + \int_{1}^{\infty} g'\varphi\,dx$$

so $$u' = (g(1)-f(1))\delta_1 + \begin{cases} f' & \text{if } x<1 \\ g' & \text{if } x>1. \end{cases}$$

<u>(c)</u> <u>(i)</u> $u' + 2xu = \delta_0$, $u \in \mathcal{D}'(\mathbb{R})$.

Multiply by $e^{x^2} \in C^\infty(\mathbb{R})$ and use Leibniz:

$$\delta_0 = e^{x^2}\delta_0 = e^{x^2}(u' + 2xu) = (e^{x^2}u)'.$$

Recall that $H' = \delta_0$, so $0 = (e^{x^2}u - H)'$.

By the constancy result from (a) we find $e^{x^2}u - H = c$, $c \in \mathbb{R}$ a constant, thus GS in $\mathcal{D}'(\mathbb{R})$ is $u = ce^{-x^2} + H(x)e^{-x^2}$, $(c \in \mathbb{R})$.

<u>(ii)</u> Let $f \in \mathcal{D}(\mathbb{R})$. $v' + 2xv = f$ in $\mathcal{D}'(\mathbb{R})$

As in (i) we multiply by $e^{x^2} \in C^\infty(\mathbb{R})$ and use Leibniz:

$$(e^{x^2}v)' = e^{x^2}f.$$

Define $F(x) = \int_0^x f(t)e^{t^2}\,dt$, $x \in \mathbb{R}$. By (standard) FTC, $F \in C^1(\mathbb{R})$ with $F'(x) = e^{x^2}f$ (so in fact $F \in C^\infty(\mathbb{R})$) and we get $(e^{x^2}v - F)' = 0$.

As above we conclude that GS is

$$v = ce^{-x^2} + F(x)e^{-x^2}, \quad (c \in \mathbb{R})$$

(iii) $w' - 2x\,w = 0$, $w \in \mathcal{S}'(\mathbb{R})$.

We proceed as above, this time multiplying by $e^{-x^2} \in C^\infty(\mathbb{R})$, and solving in $\mathcal{D}'(\mathbb{R})$ first:

$(w e^{-x^2})' = 0$, so $w = c\, e^{x^2}$, $c \in \mathbb{R}$.

Because $e^{x^2} \notin \mathcal{S}'(\mathbb{R})$ we have that $w = 0$ is GS in $\mathcal{S}'(\mathbb{R})$.

(2) $r>0$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$

r-dilation of φ is $(d_r\varphi)(x) := \varphi(rx)$, $x \in \mathbb{R}^n$.

(a) (i) Adjoint identity scheme: for $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} d_r\varphi\, \psi \, dx \underset{dy = r^n dx}{\overset{y = rx}{=}} \int_{\mathbb{R}^n} \varphi(y)\psi\left(\frac{y}{r}\right) r^{-n}\, dy$$

and so for $u \in \mathcal{D}'(\mathbb{R}^n)$ we define $d_r u$ by

$$\langle d_r u, \varphi \rangle = \langle u, \frac{1}{r^n} d_{\frac{1}{r}}\varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

We check that $d_r u \in \mathcal{D}'(\mathbb{R}^n)$.

(ii) $-n < \alpha < \infty$ and $u_\alpha(x) = |x|^\alpha$. Then by polar coordinates integration $u_\alpha \in L^1_{loc}(\mathbb{R}^n)$ so $u \in \mathcal{D}'(\mathbb{R}^n)$.

Since $\langle d_r u_\alpha, \varphi \rangle = \langle u_\alpha, \frac{1}{r^n} d_{\frac{1}{r}}\varphi \rangle =$

$$\int_{\mathbb{R}^n} |x|^\alpha \frac{1}{r^n}\varphi\left(\frac{x}{r}\right) dx \overset{y=\frac{x}{r}}{=} \int_{\mathbb{R}^n} |ry|^\alpha \varphi(y)\, dy$$

$$= r^\alpha \int_{\mathbb{R}^n} |y|^\alpha \varphi(y)\, dy = \langle r^\alpha u_\alpha, \varphi \rangle$$

$\forall \varphi \in \mathbb{R}^n$, so $d_r u_\alpha = r^\alpha u_\alpha$.

(iii) δ_0 at $0 \in \mathbb{R}^n$ is $-n$ homogeneous:

$$\langle d_r \delta_0, \varphi \rangle = \langle \delta_0, \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle = \frac{1}{r^n}\varphi(0)$$

$$= \langle \frac{1}{r^n}\delta_0, \varphi \rangle \quad \text{so} \quad d_r\delta_0 = r^{-n}\delta_0.$$

(b) For each $\varphi \in \mathcal{D}(\mathbb{R})$ put

$$\langle pv(\tfrac{1}{x}), \varphi \rangle := \lim_{a \to 0+} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{x}\, dx .$$

Fix $\varphi \in \mathcal{D}(\mathbb{R})$, say $\mathrm{supp}(\varphi) \subseteq [-A, A]$.

Then for $0 < a < A$, since $\frac{\varphi(x)}{x} \in L^1_{loc}(\mathbb{R} \setminus \{0\})$

$$\left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{x}\, dx = \left(\int_{-A}^{-a} + \int_{a}^{A} \right) \frac{\varphi(x) - \varphi(0)}{x}\, dx \overset{FTC}{=}$$

$$\left(\int_{-A}^{-a} + \int_{a}^{A} \right) \int_0^1 \varphi'(tx)\, dt\, dx .$$ Now

$$x \mapsto \int_0^1 \varphi'(tx)\, dt \in L^1(\mathbb{R})$$ clearly

so the limit $a \to 0+$, $A \to \infty$ exists and

$$\langle pv(\tfrac{1}{x}), \varphi \rangle = \int_{\mathbb{R}} \int_0^1 \varphi'(tx)\, dt\, dx$$

is well-defined. Clearly $pv(\frac{1}{x}) : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ is then linear and if $K \subset \mathbb{R}$ cpt, $\varphi \in \mathcal{D}(K)$, then with $B = \max\{|y| : y \in K\}$,

$$|\langle pv(\tfrac{1}{x}), \varphi \rangle| \leq \int_{-B}^{B} \int_0^1 |\varphi'(tx)|\, dt\, dx$$

$$\leq 2B \max |\varphi'|$$

so $pv(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R})$ (of order ≤ 1).

Next, $\langle d_r\, pv(\frac{1}{x}), \varphi \rangle = \langle pv(\frac{1}{x}), \frac{1}{r} d_{\frac{1}{r}} \varphi \rangle$

$$= \int_{\mathbb{R}} \int_0^1 \left(\tfrac{1}{r} d_{\frac{1}{r}} \varphi\right)'(tx)\, dt\, dx =$$

$$y = \frac{x}{r}$$

$$\int_{\mathbb{R}} \int_0^1 \frac{1}{r^2} \varphi'\left(\frac{tx}{r}\right) dt\, dx \underset{dy = \frac{dx}{r}}{=} \int_{\mathbb{R}} \int_0^1 \frac{1}{r} \varphi'(ty)\, dt\, dy$$

$$= \left\langle \frac{1}{r} \operatorname{pv}\left(\frac{1}{x}\right), \varphi \right\rangle \qquad \therefore \quad d_r \operatorname{pv}\left(\frac{1}{x}\right) = \frac{1}{r} \operatorname{pv}\left(\frac{1}{x}\right).$$

Finally $\log|x| \in L^1_{loc}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ and for $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subset [-A, A]$,

$$\langle (\log|x|)', \varphi \rangle = \langle \log|x|, -\varphi' \rangle =$$

$$-\int_{-\infty}^{\infty} \log|x|\, \varphi'(x)\, dx = -\int_{-\infty}^{0} \log|x|\, \varphi'(x)\, dx - \int_0^{\infty} \log|x|\, \varphi'(x)\, dx$$

$$\underset{\text{DCT}}{=} -\lim_{a \to 0+} \left(\int_{-A}^{-a} + \int_a^A \right) \log|x|\, \varphi'(x)\, dx \underset{\text{parts}}{=}$$

$$-\lim_{a \to 0+} \Big\{ \Big[\log|x| (\varphi(x) - \varphi(0)) \Big]_{x=-A}^{x=-a} - \int_{-A}^{-a} \frac{1}{x} (\varphi(x) - \varphi(0))\, dx$$

$$+ \Big[\log|x| (\varphi(x) - \varphi(0)) \Big]_{x=a}^{x=A} - \int_a^A \frac{1}{x} (\varphi(x) - \varphi(0))\, dx$$

$$= \lim_{a \to 0+} \left\{ \log a\, (\varphi(a) - \varphi(-a)) + \left(\int_{-A}^{-a} + \int_a^A \right) \frac{\varphi(x) - \varphi(0)}{x}\, dx \right\}$$

$$= \left\langle \operatorname{pv}\left(\frac{1}{x}\right), \varphi \right\rangle \qquad \therefore \quad (\log|x|)' = \operatorname{pv}\left(\frac{1}{x}\right).$$

(c) Show $u = pv(\frac{1}{x})$ solves $xu = 1$ in $\mathcal{D}'(\mathbb{R})$: 4/5

For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle x\, pv(\tfrac{1}{x}), \varphi \rangle = \langle pv(\tfrac{1}{x}), x\varphi \rangle =$$

$$\lim_{a \to 0+} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{x\varphi(x)}{x}\, dx = \int_{-\infty}^{\infty} \varphi\, dx$$

$$\therefore \quad x\, pv(\tfrac{1}{x}) = 1 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Eq is linear and $pv(\frac{1}{x})$ is PI.

GS to $xv = 0$ in $\mathcal{D}'(\mathbb{R})$: If $\varphi \in \mathcal{D}(\mathbb{R})$ then

$$\psi(x) = \begin{cases} \dfrac{\varphi(x) - \varphi(0)}{x}\,\theta(x) & x \neq 0 \\ \varphi'(0) & x = 0 \end{cases}$$

where $\theta \in \mathcal{D}(\mathbb{R})$ and $\theta = 1$ on $\mathrm{supp}(\varphi)$, is in $\mathcal{D}(\mathbb{R})$. Now for $x \neq 0$,

$$x\psi(x) = (\varphi(x) - \varphi(0))\theta(x) = \varphi(x) - \varphi(0)\theta(x),$$

or: $\varphi(x) = \varphi(0)\theta(x) + x\psi(x)$. By cont. this remains true at $x = 0$. Now

~~$\langle xv, \psi \rangle = \langle v,$~~

$$0 = \langle xv, \psi \rangle = \langle v, x\psi \rangle = \langle v, \varphi - \varphi(0)\theta \rangle$$

so $\langle v, \varphi \rangle = \langle v, \theta \rangle \varphi(0)$ $\therefore$ $v = c\delta_0$, $(c := \langle v, \theta \rangle)$.

GS : $V = c\delta_0 + pv(\frac{1}{x})$, $c \in \mathbb{R}$.

③ <u>(a)</u> (i) $\varphi \in \mathcal{S}(\mathbb{R})$ if $\varphi \in C^\infty(\mathbb{R})$ and $\overline{S}_{k,\ell}(\varphi) < \infty$ for all $k, \ell \in \mathbb{N}_0$. 1/7

$\varphi_j \to \varphi$ in $\mathcal{S}(\mathbb{R})$ if $\overline{S}_{k,\ell}(\varphi_j - \varphi) \xrightarrow[j]{} 0$ for all $k, \ell \in \mathbb{N}_0$.

$u \in \mathcal{S}'(\mathbb{R})$ if $u : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is linear and u is $\mathcal{S}$-cont: $\exists\, c \geq 0,\ k, \ell \in \mathbb{N}_0$ so $|\langle u, \varphi \rangle| \leq c\, \overline{S}_{k,\ell}(\varphi)$ $\forall \varphi \in \mathcal{S}(\mathbb{R})$.

For $f \in L^1(\mathbb{R})$ define $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} dx$.
(Clearly well-defined for each $\xi \in \mathbb{R}$ since $|f(x) e^{-ix\xi}| = |f(x)| \in L^1$).

Let $\varphi \in \mathcal{S}(\mathbb{R})$, so $\varphi \in C^\infty(\mathbb{R})$ (and so in particular measurable) and $\overline{S}_{2,0}(\varphi) < \infty$. Now $(1+x^2)|\varphi(x)| \leq 2\, \overline{S}_{2,0}(\varphi)$ $\forall x$ (from def. of $\overline{S}_{2,0}$) hence $|\varphi(x)| \leq \dfrac{2\overline{S}_{2,0}(\varphi)}{1+x^2} \in L^1(\mathbb{R})$ and so $\varphi \in L^1(\mathbb{R})$ with $\|\varphi\|_{L^1} \leq \underbrace{2\int_{\mathbb{R}} \frac{dx}{1+x^2}}_{=2\pi} \overline{S}_{2,0}(\varphi)$

<u>FIF in $\mathcal{S}(\mathbb{R})$:</u> $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is linear, bijective and $\mathcal{S}$-cont. Furthermore, for $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}^{-1}(\varphi)(x) = (2\pi)^{-1} \int_{\mathbb{R}} \hat{\varphi}(\xi) e^{ix\xi} d\xi.$$

<u>The boundedness property of $\mathcal{F}$ on $\mathcal{S}$:</u>

$$\bar{S}_{k,\ell}(\hat{\varphi}) \leq c(k,\ell)\, \bar{S}_{\ell+2,k}(\varphi)$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$ and $k, \ell \in \mathbb{N}_0$, where $c(k,\ell) > 0$ is a constant.

(In n dimensions: $\bar{S}_{k,\ell}(\hat{\varphi}) \leq c(k,\ell,n)\, \bar{S}_{\ell+n+1,k}(\varphi)$ for all $k,\ell \in \mathbb{N}_0$ and $\varphi \in \mathcal{S}(\mathbb{R})$.)

$\boxed{\text{Note}}$ The boundedness property of $\mathcal{F}$ on $\mathcal{S}(\mathbb{R})$ is merely a quantitative expression of $\mathcal{S}$-cont. and full marks would be given to a solution stating:

$\forall k, \ell \in \mathbb{N}_0\ \exists\ c > 0,\ p, q \in \mathbb{N}_0$ s.t.

$$\bar{S}_{k,\ell}(\hat{\varphi}) \leq c\, \bar{S}_{p,q}(\varphi) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

FIF on $\mathcal{S}'(\mathbb{R})$: $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ is linear, bijective and $\mathcal{S}'$-cont. Furthermore,

$$\mathcal{F}^{-1} = (2\pi)^{-1} \tilde{\mathcal{F}} \quad \text{with } \check{\varphi}(x) := \varphi(-x).$$

Here we defined $\hat{u}$ for $u \in \mathcal{S}'(\mathbb{R})$ by

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

Clearly well-defined and $\hat{u} \in \mathcal{S}'(\mathbb{R})$ since $\hat{\varphi} \in \mathcal{S}(\mathbb{R})$ and the boundedness property for $\mathcal{F}$ on $\mathcal{S}(\mathbb{R})$.

$\boxed{\text{Pf.}}$ BW see Lecture Notes.

We have $\sum_{k=1}^{\infty} u_k = u$ in $\mathcal{S}'(\mathbb{R})$ if
$\langle \sum_{k=1}^{n} u_k , \varphi \rangle \xrightarrow[n]{} \langle u, \varphi \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R})$.

For $\varphi \in \mathcal{S}(\mathbb{R})$ we have by boundedness property of $\mathcal{F}$: $\bar{S}_{2,0}(\hat{\varphi}) \leq c\,\bar{S}_{2,2}(\varphi)$ and so $|\xi^2 \hat{\varphi}(\xi)| \leq c\,\bar{S}_{2,2}(\varphi)$ $\forall \xi \in \mathbb{R}$. In particular with $\xi = k \in \mathbb{Z}\setminus\{0\}$ we get $|\hat{\varphi}(k)| \leq \frac{c\,\bar{S}_{2,2}(\varphi)}{k^2}$.

Since $e^{ikx} \in L^{\infty}(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ we get for $\varphi \in \mathcal{S}(\mathbb{R})$:
$$\langle \sum_{k=1}^{n} e^{ikx}, \varphi \rangle = \sum_{k=1}^{n} \int_{\mathbb{R}} e^{ikx} \varphi(x)\,dx =$$
$\sum_{k=1}^{n} \hat{\varphi}(-k)$ and since $|\hat{\varphi}(-k)| \leq \frac{c\,\bar{S}_{2,2}(\varphi)}{k^2}$ the series is absolutely convergent:
$$\langle \sum_{1}^{\infty} e^{ikx}, \varphi \rangle = \lim_{n\to\infty} \sum_{k=1}^{n} \hat{\varphi}(-k).$$

Clearly $\sum_{1}^{\infty} e^{ikx} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is well-def. and linear. Also
$$|\langle \sum_{1}^{\infty} e^{ikx}, \varphi \rangle| \leq \sum_{1}^{\infty} |\hat{\varphi}(-k)| \leq \sum_{1}^{\infty} \frac{c\,\bar{S}_{2,2}(\varphi)}{k^2}$$
$$= c'\,\bar{S}_{2,2}(\varphi) \qquad \left(c' = c\sum_{1}^{\infty} \frac{1}{k^2} = c\,\frac{\pi^2}{6} \right)$$
so $\sum_{1}^{\infty} e^{ikx} \in \mathcal{S}'(\mathbb{R})$. The procedure for $\sum_{1}^{\infty} \delta_k$ is similar: $|\varphi(k)| \leq \frac{\bar{S}_{2,0}(\varphi)}{k^2}$ $\forall k \in \mathbb{Z}\setminus\{0\}$ so $\sum_{1}^{\infty} \delta_k : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ well-defined and linear

Since $\left|\langle \sum_1^\infty \delta_k, \varphi \rangle\right| \le \sum_1^\infty \frac{\bar{S}_{2,0}(\varphi)}{k^2} = \frac{\pi^2}{6}\bar{S}_{2,0}(\varphi)$ 4/7

$\sum_1^\infty \delta_k \in \mathcal{S}'(\mathbb{R})$.

(iii) Denote $T_\varepsilon = \frac{1}{x+i\varepsilon} - \frac{1}{x-i\varepsilon}$. Since

$T_\varepsilon = \frac{-2i\varepsilon}{x^2+\varepsilon^2} \in L^1(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ and

$\mathrm{Log}(x+i\varepsilon) - \mathrm{Log}(x-i\varepsilon) = \frac{1}{2}\log(x^2+\varepsilon^2) + i\,\mathrm{Arg}(x+i\varepsilon)$

$- \frac{1}{2}\log(x^2+\varepsilon^2) - i\,\mathrm{Arg}(x-i\varepsilon) = i(\mathrm{Arg}(x+i\varepsilon) - \mathrm{Arg}(x-i\varepsilon))$

$\in L^\infty(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ with (C^1 functions)

$\frac{d}{dx}\left(\mathrm{Log}(x+i\varepsilon) - \mathrm{Log}(x-i\varepsilon)\right) = T_\varepsilon$ in $\mathcal{S}'(\mathbb{R})$.

Now for $\varphi \in \mathcal{S}(\mathbb{R})$: $\langle \mathrm{Log}(x+i\varepsilon) - \mathrm{Log}(x-i\varepsilon), \varphi \rangle =$

$i\int_{-\infty}^{\infty} \left(\mathrm{Arg}(x+i\varepsilon) - \mathrm{Arg}(x-i\varepsilon)\right)\varphi(x)\,dx \xrightarrow[\varepsilon\to 0+]{}$

$2\pi i \int_{-\infty}^{0} \varphi\,dx = \langle 2\pi i \check{H}, \varphi \rangle$.

Since $\frac{d}{dx}$ is $\mathcal{S}'$-cont. we get

$T_\varepsilon \xrightarrow[\varepsilon\to 0+]{} \frac{d}{dx}(2\pi i \check{H}) = -2\pi i \delta_0$. in $\mathcal{S}'(\mathbb{R})$.

(b) By def. for $\varphi \in \mathcal{S}(\mathbb{R})$: $\langle \hat{\delta}_k, \varphi \rangle =$

$\langle \delta_k, \hat{\varphi} \rangle = \hat{\varphi}(k) = \int_{\mathbb{R}} \varphi(x) e^{-ikx}\,dx$ so

$\hat{\delta}_k = e^{-ikx}$. Hence $\mathcal{F}\left(\sum_1^\infty \delta_k\right) = \left(\sum_1^\infty e^{ikx}\right)^\vee$

For $\varepsilon > 0$, $|e^{-\varepsilon \pm ix}| = e^{-\varepsilon} < 1$, so by geometric series:

$$\frac{1}{e^{\varepsilon - ix} - 1} = \frac{e^{-\varepsilon + ix}}{1 - e^{-\varepsilon + ix}} = \sum_{k=1}^{\infty} e^{(-\varepsilon + ix)k},$$

$$\frac{1}{1 - e^{-\varepsilon - ix}} = \sum_{k=0}^{\infty} e^{-(\varepsilon + ix)k} \quad \text{uniformly in } x$$

and so for $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\left\langle \frac{1}{e^{\varepsilon - ix} - 1}, \varphi \right\rangle = \sum_{1}^{\infty} \langle e^{(-\varepsilon + ix)k}, \varphi \rangle,$$

$$\left\langle \frac{1}{1 - e^{-\varepsilon - ix}}, \varphi \right\rangle = \sum_{0}^{\infty} \langle e^{-(\varepsilon + ix)k}, \varphi \rangle.$$

Let $\delta > 0$. Take $N \in \mathbb{N}$ so $\sum_{|k| > N} |\langle e^{ikx}, \varphi \rangle| < \delta$.

Then

$$\left| \left\langle \frac{1}{e^{\varepsilon - ix} - 1} - \sum_{1}^{\infty} e^{ikx}, \varphi \right\rangle \right| \leq$$

$$\sum_{k > N} |\langle e^{(-\varepsilon + ix)k} - e^{ixk}, \varphi \rangle| + \sum_{k=1}^{N} |\langle (e^{-\varepsilon k} - 1) e^{ixk}, \varphi \rangle|$$

$$\leq \sum_{k > N} (1 - e^{-\varepsilon k}) |\langle e^{ixk}, \varphi \rangle| + (1 - e^{-\varepsilon N}) \sum_{k=1}^{N} |\langle e^{ixk}, \varphi \rangle|$$

$$< \delta + \underbrace{(1 - e^{-\varepsilon N}) \sum_{k=1}^{N} |\langle e^{ixk}, \varphi \rangle|}_{< \delta}.$$

for $\varepsilon > 0$ small enough. The same argument applies to the other sum.

(c) Put $S_\varepsilon = \frac{1}{1-e^{-\varepsilon - ix}} - \frac{1}{1-e^{\varepsilon - ix}}$ and take $\varphi \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subset (-2\pi, 2\pi)$.

From HINT : $S_\varepsilon = -\frac{1}{-\varepsilon - ix} + h(-\varepsilon - ix)$

$|x| < 2\pi$ $\qquad + \frac{1}{\varepsilon - ix} - h(\varepsilon - ix)$

$= \frac{i}{x+i\varepsilon} - \frac{i}{x-i\varepsilon} + h(-\varepsilon - ix) - h(\varepsilon - ix)$

$= i T_\varepsilon + h(-\varepsilon - ix) - h(\varepsilon - ix)$, so

$\langle S_\varepsilon, \varphi \rangle = i \langle T_\varepsilon, \varphi \rangle + \langle h(-\varepsilon - ix) - h(\varepsilon - ix), \varphi \rangle$

$\xrightarrow[\varepsilon \to 0+]{} i(-2\pi i \varphi(0)) + 0 = 2\pi \varphi(0)$

by (a)(iii) and since $h(-\varepsilon - ix) - h(\varepsilon - ix) \xrightarrow[\varepsilon \to 0+]{} 0$ uniformly in $x \in \mathrm{supp}(\varphi)$.

To conclude we note that S_ε is 2π periodic and so for $\varphi \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subset (2\pi(k-1), 2\pi(k+1))$ for $k \in \mathbb{Z}$ we get $\langle S_\varepsilon, \varphi \rangle \xrightarrow[\varepsilon \to 0+]{} 2\pi \varphi(2\pi k)$.

Now by localization of distributions we then find for $\varphi \in \mathcal{S}(\mathbb{R})$

$$\langle S_\varepsilon, \varphi \rangle \xrightarrow[\varepsilon \to 0+]{} 2\pi \langle \sum_{k \in \mathbb{Z}} \delta_{2\pi k}, \varphi \rangle$$

and consequently, from (b),

$$\sum_{k \in \mathbb{Z}} e^{ikx} = 2\pi \sum_{k \in \mathbb{Z}} \delta_{2\pi k} .$$

'Localization of distributions' without further comment is sufficient in this question. One may elaborate as follows:

We know that $\sum_k e^{ikx},\ 2\pi \sum_k \delta_{2\pi k} \in \mathcal{S}'(\mathbb{R})$ so to prove that they are equal it suffices to show that

$$\langle \sum_k e^{ikx}, \varphi \rangle = \langle 2\pi \sum_k \delta_{2\pi k}, \varphi \rangle$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$ (the latter is dense in $\mathcal{S}(\mathbb{R})$).

Fix $\varphi \in \mathcal{D}(\mathbb{R})$ and note that $\{(2\pi(k-1), 2\pi(k+1)) : k \in \mathbb{Z}\}$ is an open cover of $\mathbb{R}$ and so in particular of $\mathrm{supp}(\varphi)$: for some $N \in \mathbb{N}$,

$$\mathrm{supp}(\varphi) \subset \bigcup_{k=-N}^{k=+N} (2\pi(k-1), 2\pi(k+1)).$$

Let $\{\theta_j\}_{j\in J}$ be a smooth partition of unity subordinated $(2\pi(k-1), 2\pi(k+1))$, $|k| \le N$, and so $\sum_{j\in J} \theta_j = 1$ on $\mathrm{supp}(\varphi)$. (See Theorem 2.8 in LN). Now $\varphi = \sum_{j\in J} \varphi\theta_j$ and

$$\langle \sum_k e^{ikx}, \varphi \rangle = \sum_{j\in J} \langle \sum_k e^{ikx}, \varphi\theta_j \rangle = \sum_{j\in J} \langle \sum_k 2\pi\delta_{2\pi k}, \theta_j\varphi \rangle$$

$$= \langle \sum_k 2\pi \delta_{2\pi k}, \varphi \rangle. \quad \square$$