6.1 EXX continued from L6 ...

(3) <u>Translation</u>. $T = \tau_h$, $h \in \mathbb{R}^n$, defined as $(\tau_h \varphi)(x) := \varphi(x+h)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$.
Adjoint identity with $S = \tau_{-h}$, clearly $\mathcal{D}(\mathbb{R}^n)$-continuous, and so for $u \in \mathcal{D}'(\mathbb{R}^n)$ we define $\tau_h u \in \mathcal{D}'(\mathbb{R}^n)$ by rule

$$\langle \tau_h u, \varphi \rangle \overset{\text{def}}{:=} \langle u, \tau_{-h} \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

<u>Dilation</u>. $T = d_r$, $r > 0$, defined as $(d_r \varphi)(x) := \varphi(rx)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$.
Adjoint identity with $S = \frac{1}{r^n} d_{\frac{1}{r}}$, clearly $\mathcal{D}(\mathbb{R}^n)$-continuous, and so for $u \in \mathcal{D}'(\mathbb{R}^n)$ we define $d_r u \in \mathcal{D}'(\mathbb{R}^n)$ by rule

$$\langle d_r u, \varphi \rangle \overset{\text{def}}{:=} \langle u, \tfrac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

<u>Reflection through origin</u>. $T = \widetilde{(\cdot)}$, defined as $\tilde{\varphi}(x) := \varphi(-x)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$.
Adjoint identity with $S = T$, clearly $\mathcal{D}(\mathbb{R}^n)$-continuous, and so for $u \in \mathcal{D}'(\mathbb{R}^n)$ we define $\tilde{u} \in \mathcal{D}'(\mathbb{R}^n)$ by rule

$\langle \tilde{u}, \varphi \rangle \overset{\text{def}}{=} \langle u, \tilde{\varphi} \rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Convolution with test function

Define for $\psi \in \mathcal{D}(\mathbb{R}^n)$, $T(\varphi) := \varphi * \psi$ $(= \psi * \varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Adjoint identity with $S(\varphi) = \varphi * \tilde{\psi}$ (from Fubini), clearly $\mathcal{D}(\mathbb{R}^n)$-continuous, and so for $u \in \mathcal{D}'(\mathbb{R}^n)$ we define $u * \psi$ $(= \psi * u) \in \mathcal{D}'(\mathbb{R}^n)$ by rule

$$\langle u * \psi, \varphi \rangle :\overset{\text{def}}{=} \langle u, \tilde{\psi} * \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

On sheet 2 you'll show (for $n=1$):

$$u * \psi \in C^\infty(\mathbb{R}^n),$$
$$(u * \psi)(x) = \langle u, \tilde{\psi}(x - \cdot) \rangle, \quad x \in \mathbb{R}^n.$$

Convolution can be defined in more general terms — we'll return to this later.

$\boxed{\text{6.2 DEF.}}$ Let Ω be an open non-empty subset of $\mathbb{R}^n$.

(1) For $u \in \mathcal{D}'(\Omega)$ and $j \in \{1, \dots, n\}$ we

define the distributional partial derivative $D_j u = \partial_j u = \frac{\partial u}{\partial x_j} = u_{x_j} = \in \mathcal{D}'(\Omega)$ by the rule

$$\langle D_j u, \varphi \rangle \overset{\text{def}}{:=} \langle u, -D_j \varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega).$$

Note • Well-defined by adjoint identity scheme with $T = D_j$, $S = -D_j$.

In particular, note that $u \mapsto D_j u$ is a $\mathcal{D}'(\Omega)$-continuous linear map.

• Consistent as in 1-D case when $u \in C^1(\Omega)$ we have that the distributional partial derivatives equal the usual ones.

Higher order derivatives. Since for $\varphi \in \mathcal{D}(\Omega)$ we have $D_j D_k \varphi = D_k D_j \varphi$ it follows that also $D_j D_k u = D_k D_j u$ when $u \in \mathcal{D}'(\Omega)$. Hence we can also use multi-index notation for distributional derivatives. Thus for $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^n$: $D^\alpha u \in \mathcal{D}'(\Omega)$ is given by rule

$$\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega).$$

(2) For $f \in C^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, $fu \in \mathcal{D}'(\Omega)$ is defined by rule

$$\langle fu, \varphi \rangle \overset{\text{def}}{:=} \langle u, f\varphi \rangle, \quad \varphi \in \mathcal{D}(\Omega).$$

Note • Well-defined by adjoint identity scheme with $T = S = f$ (times), and as in 1-D case it's clearly consistent with usual product when $u \in L^1_{loc}(\Omega)$.

We also define $uf :\overset{\text{def}}{=} fu$.

• The map $u \mapsto fu$ is linear and $\mathcal{D}'(\Omega)$-continuous.

$\boxed{\text{EXX}}$

△1 Let $u = \mathbb{1}_H$, where $H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$.

Then $u \in L^1_{loc}(\mathbb{R}^2) \subset \mathcal{D}'(\mathbb{R}^2)$ and

$D_1 u = 0$, $\langle D_2 u, \varphi \rangle = \langle u, -D_2\varphi \rangle =$

$$-\int_H \varphi'_y(x,y)\, d(x,y) \overset{\text{Fubini}}{=} -\int_{-\infty}^{\infty}\int_0^{\infty} \varphi'_y(x,y)\,dy\,dx$$

$$\overset{\text{FTC}}{=} \int_{-\infty}^{\infty} -\left[\varphi(x,y)\right]_{y\to 0+}^{y\to\infty} dx = \int_{-\infty}^{\infty} \varphi(x,0)\,dx.$$

$$\left(= \langle (\mathcal{L}_{\mathbb{R}}) \otimes \delta_0, \varphi \rangle \right)$$

△2 Finite difference operator For $h \in \mathbb{R}\setminus\{0\}$ define $\Delta_h := \dfrac{\tau_h - I}{h}$ on $\mathcal{D}'(\mathbb{R})$ by rule

$$\langle \Delta_h u, \varphi \rangle = \langle u, \frac{\tau_{-h} - I}{h} \varphi \rangle, \ \varphi \in \mathcal{D}(\mathbb{R}).$$

Then check that $\frac{\tau_{-h} - I}{h} \varphi \xrightarrow[h \to 0]{} -\varphi'$ in $\mathcal{D}(\mathbb{R})$, and so $\Delta_h u \xrightarrow[h \to 0]{} u'$ in $\mathcal{D}'(\mathbb{R})$.

$\boxed{\text{6.4 Theorem}}$ (Leibniz' Rule)

If $u \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega)$ and $j \in \{1, \dots, n\}$, then

$$D_j(fu) = (D_j f)u + f D_j u.$$

In fact, also the Generalized Leibniz Rule holds: for $\alpha \in \mathbb{N}_0^n$

$$D^\alpha(fu) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f \, D^{\alpha - \beta} u.$$

$\boxed{\text{Pf.}}$ The proof follows by Leibniz' Rule on $\mathcal{D}(\Omega)$ and our definitions. For $\varphi \in \mathcal{D}(\Omega)$ we calculate:

$$\langle D_j(fu), \varphi \rangle = \langle u, -f D_j \varphi \rangle,$$

$$\langle (D_j f)u + f D_j u, \varphi \rangle = \langle u, (D_j f)\varphi - D_j(f\varphi) \rangle$$

$$= \langle u, -f D_j \varphi \rangle, \text{ etc. } \square$$

Recall from Prelims Analysis:

$\boxed{\text{FTC}}$ If $f \in C(a,b)$, then for $x_0 \in (a,b)$

$u(x) = \int_{x_0}^{x} f(t)\,dt$, $x \in (a,b)$, is C^1 and $u'(x) = f(x)$. Furthermore, if $v \in C^1(a,b)$ and $v' = f$ on (a,b), then $v = u + c$ for some constant $c \in \mathbb{R}$ (or $\mathbb{C}$).

$\boxed{\text{7N. Theorem}}$ FTC^d

Let $f \in \mathcal{D}'(a,b)$. Then there exists $u \in \mathcal{D}'(a,b)$ s.t. $u' = f$ in $\mathcal{D}'(a,b)$. Furthermore, if $v \in \mathcal{D}'(a,b)$ and $v' = f$ in $\mathcal{D}'(a,b)$, then $v = u + c$ for some constant $c \in \mathbb{R}$ (or $\mathbb{C}$).

$\boxed{\text{Pf.}}$ Choose $\chi \in \mathcal{D}(a,b)$ with $\int_a^b \chi(t)\,dt = 1$.

Define for $\varphi \in \mathcal{D}(a,b)$,

$$E(\varphi)(x) := \int_a^x \varphi(t)\,dt - \int_a^b \varphi(t)\,dt \int_a^x \chi(t)\,dt, \quad x \in (a,b)$$

By FTC, $E(\varphi) \in C^1(a,b)$ with

$$E(\varphi)'(x) = \varphi(x) - \int_a^b \varphi(t)\,dt\ \chi(x).$$

Hence $E(\varphi) \in C^\infty(a,b)$. Since

$\mathrm{supp}(\chi)$, $\mathrm{supp}(\varphi) \subset (a,b)$ are compact we can find $\bar a, \bar b \in (a,b)$ with $\bar a < \bar b$ and $\chi(x) = \varphi(x) = 0$ for $x \in (a,\bar a) \cup (\bar b, b)$.

Now for $x \in (a, \bar a)$ we clearly have $E(\varphi)(x) = 0$, while for $x \in (\bar b, b)$,

$$E(\varphi)(x) = \int_a^b \varphi(t)\,dt - \int_a^b \varphi(t)\,dt \int_a^b \chi(t)\,dt = 0,$$

so that $\mathrm{supp}(E(\varphi)) \subseteq [\bar a, \bar b]$ and thus $E(\varphi) \in \mathcal{D}(a,b)$. By inspection

$$E : \mathcal{D}(a,b) \to \mathcal{D}(a,b) \quad \text{is linear.}$$

It's also $\mathcal{D}(a,b)$-continuous: if $\varphi_j \to 0$ in $\mathcal{D}(a,b)$, then for some $a < \bar a < \bar b < b$,

$$\begin{cases} \mathrm{supp}(\varphi_j) \subseteq [\bar a, \bar b], \\ \sup|\varphi_j^{(k)}| \xrightarrow[j]{} 0 \end{cases} \quad \text{for each } k \in \mathbb{N}_0$$

WLOG assume that also $\mathrm{supp}(\chi) \subseteq [\bar a, \bar b]$, whereby $\mathrm{supp}\, E(\varphi_j) \subseteq [\bar a, \bar b]$ for all j too.

Now $$\sup|E(\varphi_j)| \le (\bar b - \bar a)\left(1 + \int_a^b |\chi|\,dt\right) \sup|\varphi_j| \to 0$$

and for $k \geq 1$,

$$\sup|E(\varphi_j)^{(k)}| \le \cancel{\{1} \sup|\varphi_j^{(k-1)}| + (\bar b - \bar a) \sup|\chi^{(k-1)}| \,\}\, \sup|\varphi_j|$$

$$\xrightarrow[j]{} 0, \quad \text{so } E(\varphi_j) \to 0 \text{ in } \mathcal{D}(a,b).$$

E is a left-inverse to $\frac{d}{dx}$ on $\mathcal{D}(a,b)$:

$E(\varphi') = \varphi$ for $\varphi \in \mathcal{D}(a,b)$ by FTC.

Define $u = -f \circ E$. Then $u \in \mathcal{D}'(a,b)$ and for $\varphi \in \mathcal{D}(a,b)$,

$$\langle u', \varphi \rangle = \langle u, -\varphi' \rangle = \langle -f, E(-\varphi') \rangle$$
$$= \langle f, \varphi \rangle .$$

Finally, for $v \in \mathcal{D}'(a,b)$ with $v' = f$ we put $w = v - u \in \mathcal{D}'(a,b)$. Then $w' = 0$, and so for $\varphi \in \mathcal{D}(a,b)$,

$$0 = w'(E(\varphi)) = -\langle w, E(\varphi)' \rangle =$$
$$-\langle w, \varphi - \textstyle\int_a^b \varphi \, dt \, \chi \rangle ,$$

or: $$\langle w, \varphi \rangle = \underbrace{\langle w, \chi \rangle}_{c} \int_a^b \varphi \, dt . \quad \square$$

8.1N Corollary (= Theorem 7.3 for $n=1$)

Assume $f \in C(a,b)$. If $u \in \mathcal{D}'(a,b)$ and $u' = f$ in $\mathcal{D}'(a,b)$, then $u \in C^1(a,b)$.

Pf. Choose $x_0 \in (a,b)$ and put $F(x) = \int_{x_0}^{x} f(t)\,dt$, $x \in (a,b)$. Then by FTC, $F \in C^1(a,b)$ with $F' = f$ both classically and in $\mathcal{D}'(a,b)$. By FTC^d we can then find a constant c s.t. $u = F + c \in C^1(a,b)$. □

8.2N Corollary Assume $f \in L^1_{loc}(a,b)$. If $u \in \mathcal{D}'(a,b)$ and $u' = f$ in $\mathcal{D}'(a,b)$, then for each $x_0 \in (a,b)$ there exists a constant c s.t.

$$u(x) = \int_{x_0}^{x} f(t)\,dt + c, \quad x \in (a,b).$$

(so u is a regular distribution represented by the function on the right-hand side)

<u>Remarks</u> (Not examinable) <u>TFAE:</u>

<u>I.</u> $u \in AC_{loc}(a,b)$, that is, u is <u>locally absolutely continuous on</u> (a,b): for all $[\bar{a},\bar{b}] \subset (a,b)$ and $\varepsilon > 0$

there exists a $\delta > 0$ s.t. for any finite number of nonoverlapping intervals

$[a_1, b_1], \dots, [a_k, b_k]$ in $[\bar{a}, \bar{b}]$

with $\sum_{j=1}^{k} (b_j - a_j) < \delta$ we have

$\sum_{j=1}^{k} |u(b_j) - u(a_j)| < \varepsilon$.

II. $u \in \mathcal{D}'(a,b)$ and $u' \in L^1_{loc}$

distributional derivative

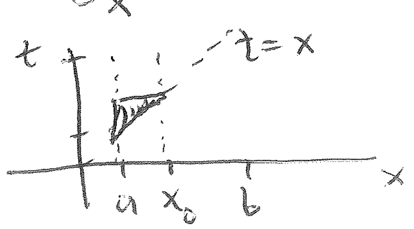
III.
$$\begin{cases} \text{(i)} & u : (a,b) \to \mathbb{R} \text{ continuous,} \\ \text{(ii)} & u \text{ differentiable a.e. in } (a,b) \\ & \text{and } u' \in L^1_{loc}(a,b), \\ \text{(iii)} & \mathcal{L}^1(u(M)) = 0 \text{ for all } M \subset (a,b) \\ & \text{with } \mathcal{L}^1(M) = 0 \text{ (Luzin (N) property)} \end{cases}$$

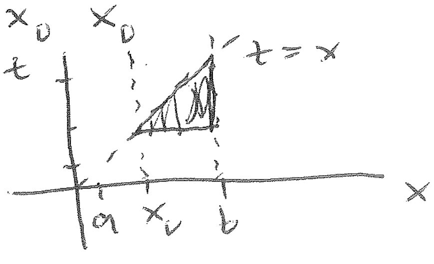
$\boxed{\text{Pf.}}$ Fix $x_0 \in (a,b)$ and put $F(x) = \int_{x_0}^{x} f(t)\,dt$, $x \in (a,b)$.

Then $F \in C(a,b)$ and for $\varphi \in \mathcal{D}(a,b)$:

$$\langle F', \varphi \rangle = \langle F, -\varphi' \rangle = -\int_a^b \int_{x_0}^{x} f(t)\,dt\,\varphi'(x)\,dx$$

$$= \int_a^{x_0} \int_x^{x_0} f(t)\varphi'(x)\,dt\,dx - \int_{x_0}^{b} \int_{x_0}^{x} f(t)\varphi'(x)\,dt\,dx$$





$\overset{\text{Fubini}}{=} \int_a^{x_0} \int_a^t f(t)\varphi'(x)\,dx\,dt - \int_{x_0}^b \int_t^b f(t)\varphi'(x)\,dx\,dt$

$\overset{\text{FTC}}{=} \int_a^b f(t)\varphi(t)\,dt$, so $F' = f$ in $\mathcal{D}'(a,b)$.

By FTC^d we have for a constant c that $u = F + c$, as required. □

Higher dimensions

$\boxed{\text{7.1 Theorem}}$ Let Ω be a nonempty open and connected subset of $\mathbb{R}^n$. If $u \in \mathcal{D}'(\Omega)$ and $D_j u = 0$ in $\mathcal{D}'(\Omega)$ for $j \in \{1, \ldots, n\}$, then there exists a constant c so $u = c$.

(ie, $\langle u, \varphi \rangle = c \int_\Omega \varphi(x)\,dx$ for $\varphi \in \mathcal{D}(\Omega)$)

Pf. omitted. □

$\boxed{\text{7.3 Theorem}}$ Let Ω be a nonempty open subset of $\mathbb{R}^n$. If $u \in \mathcal{D}'(\Omega)$ and $D_j u \in C(\Omega)$ for $j \in \{1, \ldots, n\}$, then $u \in C^1(\Omega)$.

Pf. omitted. □

What happens if $u \in \mathcal{D}'(\Omega)$ and $D_j u \in L^1_{loc}(\Omega)$ for $j \in \{1, \dots, n\}$?

We shall return to this by the end of the course, and meanwhile record a related definition:

$\boxed{\text{7.4 DEF.}}$ (Sobolev spaces)

Let Ω be a nonempty open subset of $\mathbb{R}^n$, $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then any $u \in L^p(\Omega)$ with $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq m$ is called a $W^{m,p}$ Sobolev function. The set of these, denoted

$$W^{m,p}(\Omega) := \{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq m \}$$

is called a Sobolev space. It's not hard to check that it is a vector space, and we equip it with the norm

$$\|u\|_{W^{m,p}} := \begin{cases} \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty) \\ \max_{|\alpha| \leq m} \|D^\alpha u\|_\infty & \text{if } p = \infty, \end{cases}$$

This is a norm on $W^{m,p}(\Omega)$ in the same sense that $\|\cdot\|_p$ is a norm on $L^p(\Omega)$:

on vector space of equivalence classes under the equivalence relation 'equal a.e.'.

$\boxed{\text{Remark}}$ $W^{1,1}(a,b) = L^1(a,b) \cap AC(a,b)$ in the sense that each $u \in W^{1,1}(a,b)$ has a representative (unique!) that is absolutely continuous on (a,b).

Note that when we say that a distribution $u \in \mathcal{D}'(\Omega)$ is $C(\Omega)$ it means it can be represented by such a function. Since also any $v = u$ a.e. will represent the distribution it's clear that it also has discontinuous representatives. It's well-known that we can localize functions $u \in C(\Omega)$. For instance when we say that u is C^∞ on the open subset $\omega \subset \Omega$ we mean the restriction $u|_\omega \in C^\infty(\omega)$. We can't define pointwise values of general distributions $u \in \mathcal{D}'(\Omega)$, but it's possible to localize them and it turns out that they're locally determined.

For $u \in \mathcal{D}'(\Omega)$ and an open subset $\omega \subset \Omega$ we define <u>the restriction</u> $u|_\omega$ by the rule: $\langle u|_\omega, \varphi\rangle := \langle u, \varphi\rangle$, $\varphi \in \mathcal{D}(\omega)$.

Clearly $u|_\omega \in \mathcal{D}'(\omega)$.

We say that $u \in \mathcal{D}'(\Omega)$ is 0 on ω (or is C^k on ω) iff $u|_\omega = 0$ (or $u|_\omega \in C^k(\omega)$)

Assume $u \in \mathcal{D}'(\Omega)$. If u is C^k around every point of Ω, then one may show that u is C^k on Ω.

13.10 Theorem If $u \in \mathcal{D}'(\Omega)$ and for each $x \in \Omega$ there exists $r_x > 0$ s.t. $u|_{\Omega \cap B_{r_x}(x)} = 0$, then $u = 0$.

Pf omitted. $\square$ (See lecture notes)

13.12 DEF. Let $u \in \mathcal{D}'(\Omega)$. The <u>support</u> of u is

$$\operatorname{supp}(u) \overset{\text{def}}{:=} \{x \in \Omega : u|_{\Omega \cap B_r(x)} \neq 0 \ \forall r > 0\}.$$

Thus $x \in \Omega \setminus \operatorname{supp}(u)$ iff there exists $r > 0$ so $u|_{\Omega \cap B_r(x)} = 0$. Consequently, $\Omega \setminus \operatorname{supp}(u)$ is open, and so $\operatorname{supp}(u)$ is relatively closed in Ω. From Theorem 13.10 follows that $\Omega \setminus \operatorname{supp}(u)$ is the largest open subset

$\omega \subseteq \Omega$ s.t. $u|_\omega = 0$.

$\boxed{\text{13.14 EX}}$ Definition of support is consistent with our definition of support for a continuous function.

$\boxed{\text{13.13 Theorem}}$ Let $u \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$. If $\mathrm{supp}(u) = \{x_0\}$, then $u \in \mathrm{span}\{D^\alpha \delta_{x_0} : \alpha \in \mathbb{N}_0^n\}$.

Pf is omitted. □

$\boxed{\text{Remark}}$ Let $u \in \mathcal{D}'(\Omega)$. The <u>singular support</u> of u is defined as

$$\mathrm{sing\,supp}(u) \overset{\text{def}}{:=} \{x \in \Omega : u|_{\Omega \cap B_r(x)} \notin C^\infty \ \forall r > 0\}$$

Thus $x \in \Omega \setminus \mathrm{sing\,supp}(u)$ iff there exists $r > 0$ so $u|_{\Omega \cap B_r(x)} \in C^\infty$. Consequently, $\Omega \setminus \mathrm{sing\,supp}(u)$ is open, and so $\mathrm{sing\,supp}(u)$ is relatively closed in Ω. Using a result related to Theoram 13.10 one can show that $\Omega \setminus \mathrm{sing\,supp}(u)$ is the largest open subset $\omega \subseteq \Omega$ s.t $u|_\omega \in C^\infty(\omega)$.
Clearly we have $\mathrm{sing\,supp}(u) \subseteq \mathrm{supp}(u)$.