Distribution Theory and Fourier Analysis: An Introduction MT18

Problem Sheet 1

Problem 1. Define $\phi\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} \mathrm{e}^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

Show that ϕ is C^∞, and deduce that

$$\psi(x) = \phi\big(2(1-x)\big)\phi\big(2(1+x)\big)$$

belongs to $\mathscr{D}(\mathbb{R})$. Does the restriction to $(-1,1)$, $\psi|_{(-1,1)}$, belong to $\mathscr{D}(-1,1)$?

Calculate the Taylor series for ϕ about 0 (note: not for ψ). Does the series converge, and if so, then what is its sum?

Problem 2. In this question all functions and distributions are real-valued.

(a) Let K be a compact proper subset of the open interval (a,b). Show carefully that there exists $\rho \in \mathscr{D}(a,b)$ such that $0 \le \rho \le 1$ and $\rho = 1$ on K.

(b) Give an example of φ, $\psi \in \mathscr{D}(\mathbb{R})$ such that $\max(\varphi,\psi)$, $\min(\varphi,\psi)$ are *not* smooth compactly supported test functions. Here we define $\max(\varphi,\psi)(x) = \max\{\varphi(x),\psi(x)\}$ for each x and similarly for $\min(\varphi,\psi)$.

Next, let $u \in \mathscr{D}(a,b)$. Show that there exist u_1, $u_2 \in \mathscr{D}(a,b)$ with $u_1 \ge 0$, $u_2 \ge 0$ and $u = u_1 - u_2$.

(c) Generalize the last statement to n dimensions as follows. Let Ω be a nonempty open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}(\Omega)$. Show that there exist u_1, $u_2 \in \mathscr{D}(\Omega)$ with $u_1 \ge 0$ and $u_2 \ge 0$ such that $u = u_1 - u_2$.

(*Hint: You may for instance note that* $4u = (u+1)^2 - (u-1)^2$ *and if* v *is a cut-off function between the support of* u *and the boundary of* Ω*, then* $vu = u$.)

Problem 3. Let Ω be a nonempty and open subset of $\mathbb{R}^n$, $1 \le p < \infty$ and $f \in \mathrm{L}^p(\Omega)$. Show that for each $\varepsilon > 0$ there exists $g \in \mathscr{D}(\Omega)$ such that $\|f - g\|_p < \varepsilon$.

(*Hint: One approach is to do it in two steps. First choose an appropriate open subset* $O \subset \Omega$ *so that* $h = f\mathbf{1}_O$ *is a good* L^p *approximation of* f*. Then use a result from lectures.*)

Problem 4. In each of the following 3 cases decide whether or not u_j is a distribution:

$$\langle u_1, \varphi\rangle = \sum_{j=1}^{\infty} 2^{-j}\varphi^{(j)}(0), \quad \langle u_2, \varphi\rangle = \sum_{j=1}^{\infty} 2^{j}\varphi^{(j)}(j), \quad \langle u_3, \varphi\rangle = \varphi(0)^2,$$

where $\varphi \in \mathscr{D}(\mathbb{R})$ is so that the expression makes sense.

Problem 5. Let $a > 0$. For each $\varphi \in \mathscr{D}(\mathbb{R})$ we let

$$\langle T_a, \varphi \rangle = \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{|x|} \,\mathrm{d}x + \int_{-a}^{a} \frac{\varphi(x) - \varphi(0)}{|x|} \,\mathrm{d}x.$$

Show that T_a hereby is well-defined and that it is a distribution on $\mathbb{R}$.

Now assume that $\varphi \in \mathscr{D}(\mathbb{R})$ satisfies $\varphi(0) = 0$. Show that then

$$\langle T_a, \varphi \rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{|x|} \,\mathrm{d}x.$$

What distribution is $T_a - T_b$ for $0 < b < a$?

Problem 6. Let Ω be a nonempty and open subset of $\mathbb{R}^n$. Show that if $u,\ v \in \mathscr{D}'(\Omega)$ and $\langle u, \varphi \rangle = 0$ for all $\varphi \in \mathscr{D}(\Omega)$ for which $\langle v, \varphi \rangle = 0$, then $u = cv$ for some constant $c \in \mathbb{R}$.

Problem 7. (Optional) Denote $x_+ = \max\{0, x\}$ for $x \in \mathbb{R}$ and fix $m \in \mathbb{N}_0$. Show that if

$$E(x) = \frac{x_+^m}{m!}$$

where for $m = 0$ we interpret this as the Heaviside function, then

$$\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}} E = \delta_0 \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

Problem 8. (Optional and harder)
(i) Construct $g \in \mathscr{D}(\mathbb{R})$ supported in $[-1, 1]$ such that $g(0) = 1$ and $g^{(j)}(0) = 0$ for all $j \in \mathbb{N}$. (*Hint: First find $\varphi \in \mathscr{D}(\mathbb{R})$ supported in $[0, 1]$ with $\int_{\mathbb{R}} \varphi = 1$. Then consider the solution to $y'(x) = \varphi(-x) - \varphi(x)$ with support in $[-1, 1]$.*)
(ii) Let $(a_n)_{n=0}^{\infty}$ be an arbitrary sequence of real numbers. Define for $n \in \mathbb{N}_0$ and positive numbers $\varepsilon_n > 0$ the functions

$$g_n(x) = g\left(\frac{x}{\varepsilon_n}\right) \frac{a_n x^n}{n!}, \quad x \in \mathbb{R}.$$

Check that g_n is C^∞ with support contained in $[-\varepsilon_n, \varepsilon_n]$ and $g_n^{(k)}(0) = a_n \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker delta.

Show that for each $n \in \mathbb{N}_0$ it is possible to choose $\varepsilon_n > 0$ so small that

$$|g_n^{(k)}(x)| \leq 2^{-n} \tag{1}$$

holds for all $x \in \mathbb{R}$ and each $0 \leq k < n$.
(iii) We now fix each ε_n so that (1) holds. With these choices we define

$$f(x) = \sum_{n=0}^{\infty} g_n(x), \quad x \in \mathbb{R}.$$

Check that $f \in \mathscr{D}(\mathbb{R})$ with support contained in $[-1,1]$ and that $f^{(n)}(0) = a_n$ for all $n \in \mathbb{N}_0$. (*This result is due to E. Borel.*)

Problem 9. (Optional and harder)

This problem gives an alternative construction of a smooth compactly supported test function. We start with a rough convolution kernel $h = \mathbf{1}_{(0,1)}$ and put as usual for each $r > 0$,

$$h_r(x) = \frac{1}{r} h\left(\frac{x}{r}\right) = \frac{1}{r}\mathbf{1}_{(0,r)}(x), \quad x \in \mathbb{R}.$$

(i) Let $0 < r \leq s$. Show that $h_r * h_s$ is continuous, $\operatorname{spt}(h_r * h_s) = [0, r+s]$ and $0 \leq h_r * h_s \leq \frac{1}{s}$.
(ii) Let $k \in \mathbb{N}_0$ and assume $u \in \mathrm{C}_c^k(\mathbb{R})$ with $\operatorname{spt}(u) \subseteq [a,b]$. Prove that $h_r * u \in \mathrm{C}_c^{k+1}(\mathbb{R})$ with $\operatorname{spt}(h_r * u) \subseteq [a, b+r]$ and

$$(h_r * u)^{(k+1)}(x) = \frac{u^{(k)}(x) - u^{(k)}(x-r)}{r}.$$

(iii) Let $(r_j)_{j=0}^{\infty}$ be a decreasing sequence of positive numbers and put

$$R_n = \sum_{j=0}^{n} r_j.$$

Define $u_n = h_{r_0} * h_{r_1} * \cdots * h_{r_n}$ for each $n \in \mathbb{N}$.

Show that $u_n \in \mathrm{C}_c^{n-1}(\mathbb{R})$ with $\operatorname{spt}(u_n) \subseteq [0, R_n]$ and

$$|u_n^{(k)}(x)| \leq \frac{2^k}{r_0 r_1 \cdots r_k}$$

for all $x \in \mathbb{R}$ and $0 \leq k < n$. (*Hint: Proceed by induction on n and write $u_n = h_{r_0} * v_n$ for some suitable v_n.*)
(iv) Assume that

$$R = \sum_{j=0}^{\infty} r_j < \infty.$$

Show that (u_n) is a uniform Cauchy sequence. By suitable iteration of this, deduce that the limit function

$$u(x) = \lim_{n \to \infty} u_n(x), \quad x \in \mathbb{R},$$

belongs to $\mathscr{D}(\mathbb{R})$ with $\operatorname{spt}(u) \subseteq [0, R]$ and

$$|u^{(k)}(x)| \leq \frac{2^k}{r_0 r_1 \cdots r_k}$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$. In particular, $u \in \mathscr{D}(\mathbb{R})$, $0 \leq u \leq \frac{1}{r_0}$ and $\int_{\mathbb{R}} u = 1$.