Distribution Theory and Fourier Analysis: An Introduction MT18

Problem Sheet 2

Problem 1. Let $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ and $T > 0$. Then we say that g is T periodic iff $g(x+T) = g(x)$ holds for almost all $x \in \mathbb{R}$. Fix an open interval (a,b) in $\mathbb{R}$, where we allow $a = -\infty$ and $b = \infty$ but require $a < b$. Assuming that g is T periodic we define for each $j \in \mathbb{N}$ the function

$$g_j(x) = g(jx), \quad x \in (a,b).$$

Prove that

$$g_j \to \frac{1}{T}\int_0^T g \,\mathrm{d}x \,\mathbf{1}_{(a,b)} \quad \text{in} \quad \mathscr{D}'(a,b) \quad \text{as} \quad j \to \infty,$$

where $\mathbf{1}_{(a,b)}$ is the indicator function for (a,b).
(*Hint: Assume first that on the interval* $(0,T]$ *the function* g *is an indicator function* $\mathbf{1}_{(c,d]}$ *where* $0 \leq c < d \leq T$. *Then use (no need to prove it) that step functions are dense in* $\mathrm{L}^1(0,T]$ *to conclude.*)

Problem 2. Let $f, g \in \mathrm{C}^1(\mathbb{R})$ and define

$$u(x) = \begin{cases} f(x) & \text{if } x < 0 \\ g(x) & \text{if } x \geq 0. \end{cases}$$

Explain why $u \in \mathscr{D}'(\mathbb{R})$ and calculate the distributional derivative u'.

Problem 3. (a) Let $\alpha \in (-n, \infty)$ and $u_\alpha(x) = |x|^\alpha$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that $u_\alpha \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$.
(*Hint: Use polar coordinates.*)
(b) For each $r > 0$ we define the r-dilation of a test function $\varphi \in \mathscr{D}(\mathbb{R}^n)$ by the rule

$$(d_r\varphi)(x) = \varphi(rx), \quad x \in \mathbb{R}^n.$$

Extend the r-dilation to distributions $u \in \mathscr{D}'(\mathbb{R}^n)$.
(c) Show that for the distribution u_α defined in (a) we have $d_r u_\alpha = r^\alpha u_\alpha$ for all $r > 0$. We express this by saying that u_α is *homogeneous of degree* α.
(d) Show that the Dirac delta function δ_0 concentrated at the origin $0 \in \mathbb{R}^n$ is homogeneous of degree $-n$.
(e) Let $u \in \mathscr{D}'(\mathbb{R}^n)$ be homogeneous of degree $\beta \in \mathbb{R}$: $d_r u = r^\beta u$ for all $r > 0$. Show that for each $j \in \{1, \ldots, n\}$ the distribution $x_j u$ is homogeneous of degree $\beta + 1$ and that the distribution $D_j u$ is homogeneous of degree $\beta - 1$. Finally show that

$$\sum_{j=1}^n x_j D_j u = \beta u.$$

Problem 4. (a) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is piecewise continuous and $k \in \mathbb{R}$, then the function $u(x,t) = f(x-kt)$, $(x,t) \in \mathbb{R}^2$, is locally integrable on $\mathbb{R}^2$. Conclude that it defines a distribution and that it satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

in the sense of distributions on $\mathbb{R}^2$.
(b) Prove that $u(x,y) = \log\bigl(x^2+y^2\bigr)$ is locally integrable on $\mathbb{R}^2$, and that we have

$$\Delta u = 4\pi \delta_0$$

in the sense of distributions on $\mathbb{R}^2$, where δ_0 is the Dirac delta function on $\mathbb{R}^2$ concentrated at the origin.

Problem 5. Show that $u = \delta_0$, the Dirac delta function concentrated at $0 \in \mathbb{R}$, satisfies the equation

$$xu = 0. \tag{1}$$

Find the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (1). (*Hint: Show first that if $\eta \in \mathscr{D}(\mathbb{R})$ and $\eta(0) = 1$, then for any $\varphi \in \mathscr{D}(\mathbb{R})$ we can find $\psi \in \mathscr{D}(\mathbb{R})$ and $c \in \mathbb{R}$ such that $\varphi = x\psi + c\eta$. Then proceed as in a proof from lectures.*)

Problem 6. Let $\theta \in \mathscr{D}(\mathbb{R})$.
(i) Explain how the convolution $\theta * u$ is defined for a general distribution $u \in \mathscr{D}'(\mathbb{R})$.
(ii) Prove that $\theta * u \in \mathrm{C}^\infty(\mathbb{R})$ when $u \in \mathscr{D}'(\mathbb{R})$.
(iii) Let $\bigl(\rho_\varepsilon\bigr)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}$. Show that for a general distribution $u \in \mathscr{D}'(\mathbb{R})$ we have that

$$\rho_\varepsilon * u \to u \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } \varepsilon \to 0^+.$$

(iv) Show that for each $u \in \mathscr{D}'(\mathbb{R})$ we can find a sequence (u_j) in $\mathscr{D}(\mathbb{R})$ such that

$$u_j \to u \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } j \to \infty.$$

Problem 7. (Optional)
Define for each $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle \mathrm{pv}\bigl(\frac{1}{x}\bigr), \varphi\rangle = \lim_{a \to 0^+} \left(\int_{-\infty}^{-a} + \int_a^\infty \right) \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

(a) Show that hereby $\mathrm{pv}\bigl(\frac{1}{x}\bigr) \in \mathscr{D}'(\mathbb{R})$ and that it is homogeneous of order -1 (see Problem 3). Check that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log|x| = \mathrm{pv}\bigl(\frac{1}{x}\bigr).$$

(b) Show that $u = \mathrm{pv}\bigl(\frac{1}{x}\bigr)$ solves the equation

$$xu = 1 \tag{2}$$

in the sense of $\mathscr{D}'(\mathbb{R})$. What is the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (2)?