Distribution Theory and Fourier Analysis: An Introduction MT18

Problem Sheet 3

Problem 1. Let
$$p(D) = \sum_{|\alpha| \leq k} c_\alpha D^\alpha$$
be a partial differential operator on $\mathbb{R}^n$ in the usual multi-index notation. For an open subset Ω of $\mathbb{R}^n$ and $u \in \mathcal{D}'(\Omega)$ show that the supports always obey the rule:
$$\operatorname{supp}\bigl(p(D)u\bigr) \subseteq \operatorname{supp}(u).$$
Give an example of a distribution $v \in \mathcal{D}'(\mathbb{R})$ such that the distributional derivative $v' \neq 0$ has compact support, but v itself hasn't.

Problem 2. Let $f \in \mathrm{L}^1(\mathbb{R}^n)$ and denote by $(e_j)_{j=1}^n$ the standard basis for $\mathbb{R}^n$. For $\xi \in \mathbb{R}^n$ we write $\xi = \xi_1 e_1 + \cdots + \xi_n e_n$. Show that if $\xi_j \neq 0$, then
$$\hat{f}(\xi) = -\int_{\mathbb{R}^n} f\bigl(x + \frac{\pi}{\xi_j} e_j\bigr) \mathrm{e}^{-ix\cdot\xi} \, \mathrm{d}x,$$
and conclude that
$$\bigl|\hat{f}(\xi)\bigr| \leq \frac{1}{2} \int_{\mathbb{R}^n} \bigl|f(x) - f\bigl(x + \frac{\pi}{\xi_j} e_j\bigr)\bigr| \, \mathrm{d}x.$$
Using that $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathrm{L}^1(\mathbb{R}^n)$ deduce the *Riemann-Lebesgue Lemma*:
$$\hat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty.$$

Problem 3. Let $t > 0$ and put $G_t(x) = \mathrm{e}^{-t|x|^2}$ for $x \in \mathbb{R}^n$. Use the Fourier transform to find a formula for the convolution $G_s * G_t$ for all $s, t > 0$.

Problem 4. Let $a > 0$ and $b, c \in \mathbb{R}$. Put $g(x) = \mathrm{e}^{-ax^2+bx+c}$, $x \in \mathbb{R}$. Calculate $\hat{g}$.

Problem 5.
(a) State the Fourier Inversion Formula for the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. Define the Fourier transform $\hat{u}$ for a tempered distribution u on $\mathbb{R}^n$ and deduce the Fourier Inversion Formula for $\mathcal{S}'(\mathbb{R}^n)$ from the one on the Schwartz class.
(b) Derive the *Fourier-Gel'fand formula* for the Dirac delta-function at x on $\mathbb{R}^n$:
$$\delta_x = \lim_{j\to\infty} (2\pi)^{-n} \int_{(-j,j)^n} \mathrm{e}^{\mathrm{i}(\cdot - x)\cdot\xi} \, \mathrm{d}\xi \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

(c) Explain why a polynomial $p(x)$ on $\mathbb{R}$ is a tempered distribution and calculate its Fourier transform $\hat{p}$. What can you say about a tempered distribution $u \in \mathcal{S}'(\mathbb{R})$ whose Fourier transform $\hat{u}$ has $\mathrm{supp}(\hat{u}) = \{0\}$? (*Hint: Use Theorem 13.13 from the Lecture Notes.*)
(d) Show that $f \in \mathrm{L}^1(\mathbb{R}^n)$ is real-valued if and only if $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$ holds for all $\xi \in \mathbb{R}^n$. What is the corresponding result for general tempered distributions?
(e) Show that $\mathcal{F}^4(\varphi) = (2\pi)^{2n}\varphi$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. What does this say about the possible eigenvalues for $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$? Generalize your result to the Fourier transform on the tempered distributions.

Problem 6. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution that is homogeneous of degree $\alpha \in \mathbb{R}$: $d_r u = r^\alpha u$ holds for all $r > 0$. Show that the Fourier transform $\hat{u}$ is homogeneous of degree $-n - \alpha$.

Problem 7. (Optional)
Let $f(x) = x_1 |x|^\alpha$, $x \in \mathbb{R}^n$, where the dimension $n \geq 2$ and the exponent $\alpha \in (-n-1, -2)$. Compute $\hat{f}(\xi)$.
(*Hint: Consider* $\frac{\partial}{\partial x_1}|x|^{\alpha+2}$ *and look at the proof of Lemma 13.9 in the Lecture Notes.*)

Problem 8. (Optional)
Define for each $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\langle \mathrm{pv}\big(\frac{1}{x}\big), \varphi \rangle = \lim_{a \to 0^+} \left(\int_{-\infty}^{-a} + \int_a^\infty \right) \frac{\varphi(x)}{x} \, \mathrm{d}x.$$

Show that hereby $\mathrm{pv}\big(\frac{1}{x}\big) \in \mathcal{S}'(\mathbb{R})$ and calculate its Fourier transform.

(*The map* $\varphi \mapsto \mathrm{pv}\big(\frac{1}{\pi x}\big) * \varphi$ *is called the Hilbert transform.*)