Distribution Theory and Fourier Analysis: An Introduction MT18/HT19

Problem Sheet 4

Problem 1. Prove that for every $t > 0$ and $\varphi \in \mathcal{S}(\mathbb{R})$ the identity

$$\int_{-t}^{t} \hat{\varphi}(\xi)\,\mathrm{d}\xi = 2\int_{-\infty}^{\infty} \varphi(x)\frac{\sin(tx)}{x}\,\mathrm{d}x$$

holds true. Deduce that

$$\lim_{t\to\infty} \frac{\sin(tx)}{x} = \pi\delta_0 \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

where δ_0 is Dirac's delta-function concentrated at 0 on $\mathbb{R}$. (*Hint: For instance use the Product Rule and the Fourier Inversion Formula in $\mathcal{S}'$ on the left-hand side of the identity.*)

Problem 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying $|f(x)| \leq \mathrm{e}^{-|x|}$ for almost all $x \in \mathbb{R}$. Prove that the Fourier transform $\hat{f}$ cannot have compact support unless $f(x) = 0$ for almost all $x \in \mathbb{R}$. (*Hint: Use a Differentiation Rule to see that $\hat{f}$ is C^∞ and consider a suitable Taylor expansion.*)

Problem 3. Let $f(x) = \mathrm{e}^{-|x|}$, $x \in \mathbb{R}^n$.
(a) Compute the Fourier transform $\hat{f}(\xi)$ when $n = 1$. Deduce for $\lambda \geq 0$ the identity

$$\mathrm{e}^{-\lambda} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+|\xi|^2}\mathrm{e}^{i\lambda\xi}\,\mathrm{d}\xi.$$

(b) Using $\frac{1}{1+|\xi|^2} = \int_0^\infty \mathrm{e}^{-(1+|\xi|^2)t}\,\mathrm{d}t$ and (a) show that for each $\lambda \geq 0$ the identity

$$\mathrm{e}^{-\lambda} = \int_0^\infty \frac{1}{\sqrt{\pi t}}\mathrm{e}^{-t-\frac{\lambda^2}{4t}}\,\mathrm{d}t$$

holds.
(c) Compute the Fourier transform $\hat{f}(\xi)$ in the general n-dimensional case, for instance by use of the formula from (b) with $\lambda = |x|$ and calculations similar to those done in lectures when computing fundamental solutions.

Problem 4. Define for $\alpha > 0$ the function $g(x) = \left(1+|x|^2\right)^{-\frac{\alpha}{2}}$, $x \in \mathbb{R}^n$.
(a) Explain why $g \in \mathcal{S}'(\mathbb{R}^n)$. For which values of $\alpha > 0$ is g integrable over $\mathbb{R}^n$?
(b) Show that there exists a positive constant $c = c(\alpha)$ such that

$$g(x) = c\int_0^\infty t^{\frac{\alpha}{2}-1}\mathrm{e}^{-t}\mathrm{e}^{-t|x|^2}\,\mathrm{d}t$$

holds for all $x \in \mathbb{R}^n$.

(c) Using (b) show that the Fourier transform $\hat{g}$ is a positive and integrable function on $\mathbb{R}^n$. (*The function $\hat{g}$ is called the Bessel kernel of order α.*)

Problem 5. The principal logarithm is defined on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ as

$$\mathrm{Log}z := \log|z| + \mathrm{i}\mathrm{Arg}(z), \quad \mathrm{Arg}(z) \in (-\pi, \pi).$$

Define $\mathrm{Log}(x + \mathrm{i}0)$ and $\mathrm{Log}(x - \mathrm{i}0)$ for each $\varphi \in \mathcal{S}(\mathbb{R})$ by the rules

$$\langle \mathrm{Log}(x \pm \mathrm{i}0), \varphi \rangle := \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} \mathrm{Log}(x \pm \mathrm{i}\varepsilon)\varphi(x)\,\mathrm{d}x.$$

(a) Show that $\mathrm{Log}(x \pm \mathrm{i}0)$ hereby are tempered distributions on $\mathbb{R}$, and that for some constant c the inequality

$$\left| \langle \mathrm{Log}(x \pm \mathrm{i}0), \varphi \rangle \right| \leq c\overline{S}_{2,0}(\varphi)$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R})$. Recall that for $m, n \in \mathbb{N}_0$ we defined

$$\overline{S}_{m,n}(\varphi) := \sup\Big\{ |x^r \varphi^{(s)}(x)| : r \in \{0, \dots, m\}, s \in \{0, \dots, n\}, x \in \mathbb{R} \Big\}.$$

Now let $k \in \mathbb{N}$ and define the tempered distributions $(x + \mathrm{i}0)^{-k}$ and $(x - \mathrm{i}0)^{-k}$ as

$$(x \pm \mathrm{i}0)^{-k} := \frac{(-1)^{k-1}}{(k-1)!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{Log}(x \pm \mathrm{i}0) \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

(b) Show that for each $\varphi \in \mathcal{S}(\mathbb{R})$ with $\varphi^{(j)}(0) = 0$ for $j \in \{0, \dots, k\}$ we have

$$\langle (x \pm \mathrm{i}0)^{-k}, \varphi \rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{x^k}\,\mathrm{d}x.$$

Show also that the inequality

$$\left| \langle (x \pm \mathrm{i}0)^{-k}, \varphi \rangle \right| \leq \frac{c}{(k-1)!} \overline{S}_{2,k}(\varphi)$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R})$, where c is the constant you found in (a) above.
(c) Prove that $\mathrm{Log}(x + \mathrm{i}0) - \mathrm{Log}(x - \mathrm{i}0) = 2\pi\mathrm{i}\tilde{H}$, where H is the Heaviside function. Deduce the *Plemelj-Sokhotsky jump relations*:

$$(x + \mathrm{i}0)^{-k} - (x - \mathrm{i}0)^{-k} = 2\pi\mathrm{i}\frac{(-1)^k}{(k-1)!}\delta_0^{(k-1)},$$

where δ_0 is Dirac's delta-function on $\mathbb{R}$ concentrated at 0.
(d) Show that

$$x(x \pm \mathrm{i}0)^{-1} = 1 \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Deduce that

$$\big(x+\mathrm{i}0\big)^{-1}(x\delta_0) = 0 \neq \delta_0 = \Big(\big(x+\mathrm{i}0\big)^{-1}x\Big)\delta_0.$$

Next, show, for instance by using the differential operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ on the case $k=1$ iteratively, that

$$x^k(x \pm \mathrm{i}0)^{-k} = 1 \quad \text{in } \mathcal{S}'(\mathbb{R})$$

holds for each $k \in \mathbb{N}$.
(e) Let H be the Heaviside function and define for each $\varepsilon > 0$ the function $H_\varepsilon(x) := \mathrm{e}^{-\varepsilon x}H(x)$. By first calculating $\hat{H}_\varepsilon$ show that

$$\mathcal{F}_{x\to\xi}\big(H\big) = -\mathrm{i}\big(\xi - \mathrm{i}0\big)^{-1} \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Use the Fourier Inversion Formula in $\mathcal{S}'$ to find the Fourier transform of $\big(x+\mathrm{i}0\big)^{-1}$.

Problem 6. For each $\varphi \in \mathcal{S}(\mathbb{R})$ we define its 2π *periodization* as

$$P\varphi(x) = \sum_{k\in\mathbb{Z}} \varphi(x - 2\pi k), \quad x \in \mathbb{R}.$$

(a) Check that $P\varphi$ is a 2π periodic C^∞ function, and explain why

$$P\varphi(x) = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\hat{\varphi}(k)\mathrm{e}^{ikx}$$

holds for all $x \in \mathbb{R}$. (*Hint: For the latter it suffices to quote results postulated in Prelims.*)
(b) Prove *Poisson's summation formula*:

$$2\pi \sum_{k\in\mathbb{Z}} \varphi(2\pi k) = \sum_{k\in\mathbb{Z}} \hat{\varphi}(k).$$

(c) Show that

$$\sum_{k\in\mathbb{Z}} \mathrm{e}^{-4\pi^2 t k^2} = \frac{1}{\sqrt{4\pi t}} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\frac{k^2}{4t}}$$

holds for all $t > 0$.

Problem 7. (Optional)
Let $F\colon \mathbb{C} \to \mathbb{C}$ be an entire function that is not identically zero. Explain why the formula $f = \log|F|$ defines a distribution on $\mathbb{C}$ that in general will not be tempered.
Prove that its distributional Laplacian equals

$$\Delta f = \sum_{j\in J} 2\pi m_j \delta_{z_j}$$

where $\{z_j : j \in J\}$ are the distinct zeros for F and $\{m_j : j \in J\}$ their multiplicities.