(1) ϕ clearly C^∞ on $(-\infty,0)\cup(0,\infty)$ with $\phi^{(k)}(x)=0$ for $x<0$. For $x>0$, $\phi'(x)=\frac{1}{x^2}e^{-\frac{1}{x}}$, $\phi''(x)=\frac{-2x+1}{x^4}e^{-\frac{1}{x}}$ and then claim: $\phi^{(k)}(x)=\frac{p_k(x)}{x^{2k}}e^{-\frac{1}{x}}$, $x>0$, where p_k real polynomial of degree at most $k-1$. By induction: ok for $k=1,2$; if true for k, then $\phi^{(k+1)}(x)=\frac{p_k'(x)x^2-2kp_k(x)x+p_k(x)}{x^{2(k+1)}}$

$e^{-\frac{1}{x}}$ and $p_{k+1}(x):=p_k'(x)x^2-2kp_k(x)x+p_k(x)$ is a real polynomial of degree at most $\deg(p_k)+1\leq k$.

Claim ϕ infinitely often diff. at $x=0$ with $\phi^{(k)}(0)=0$ for all k.

Induction on k: $k=1$

For $x>0$ we have by Mean Value Theorem

$$\frac{\phi(x)-\phi(0)}{x}=\phi'(\theta)\quad\text{for some } \theta\in(0,x).$$

Now $\phi'(\theta)=\frac{1}{\theta^2}e^{-\frac{1}{\theta}}$ and $e^t>\frac{t^3}{6}$ for $t>0$

so $e^{-\frac{1}{\theta}} < \frac{6}{(\frac{1}{\theta})^3} = 6\theta^3$ and

thus $\phi'(\theta) < 6\theta < x \to 0$ as $x \to 0^+$.

It follows that $\lim_{x\to 0} \frac{\phi(x)-\phi(0)}{x} = 0$ so ϕ

is diff. at 0 with $\phi'(0)=0$.

Assume ϕ is k times diff. at 0 with $\phi^{(k)}(0)=0$. Then

$$\phi^{(k)}(x) = \begin{cases} \frac{p_k(x)}{x^{2k}} e^{-\frac{1}{x}}, & x>0 \\ 0, & x \le 0. \end{cases}$$

For $x>0$ we get as above:

$\frac{1}{x}\left(\phi^{(k)}(x) - \phi^{(k)}(0)\right) = \phi^{(k+1)}(\theta)$ for a $\theta \in (0,x)$.

Note $e^t > \frac{t^{2k+3}}{(2k+3)!}$ for $t>0$ so that

$$\phi^{(k+1)}(\theta) = \frac{p_{k+1}(\theta)}{\theta^{2k+2}} e^{-\frac{1}{\theta}} < \frac{p_{k+1}(\theta)}{\theta^{2k+2}} \frac{(2k+3)!}{(\frac{1}{\theta})^{2k+3}} =$$

$(2k+3)!\, p_{k+1}(\theta)\,\theta \to 0$ as $x \to 0^+$.

It follows that $\lim_{x\to 0} \frac{\phi^{(k)}(x) - \phi^{(k)}(0)}{x} = 0$

so $\phi^{(k)}$ is diff. at 0 with $\phi^{(k+1)}(0) = 0$. □

By Leibniz ψ is C^∞, and $\psi(x) \neq 0$ iff

$2(1-x)>0$ and $2(1+x)>0$, that is, iff $-1<x<1$. Thus $\mathrm{spt}(\psi)=[-1,1]$, and therefore $\psi \in \mathcal{D}(\mathbb{R})$.

Functions in $\mathcal{D}(-1,1)$ must have compact support in $(-1,1)$, so $\psi|_{(-1,1)} \notin \mathcal{D}(-1,1)$.

Taylor series for ψ about 0 is $\sum_{n=0}^{\infty} 0 x^n = 0$.

---

(2) (a) Let $\psi \in \mathcal{D}(\mathbb{R})$ be as in (1) and put $\phi = \dfrac{\psi}{\int_{\mathbb{R}} \psi \, dx}$. Then $\phi \in \mathcal{D}(\mathbb{R})$, $\phi \geq 0$, $\mathrm{spt}(\phi) = [-1,1]$ and $\int \phi \, dx = 1$. Put $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(\varepsilon^{-1}x)$ so that $(\phi_\varepsilon)_{\varepsilon>0}$ is standard mollifier. Because $K \subset (a,b)$ is compact $d = \mathrm{dist}(K, \{a,b\}) > 0$ and we can take $\rho(x) = (\phi_\varepsilon * \mathbb{1}_{B_{\frac{d}{2}}(K)})(x)$ for $\varepsilon < \frac{d}{4}$.

By result from lectures $\rho \in C^\infty(\mathbb{R})$.

If $x \notin \overline{B_{\frac{3d}{4}}(K)}$, so $\mathrm{dist}(x, \overline{B_{\frac{d}{2}}(K)}) > \frac{d}{4}$, then $\rho(x) = \int_{\overline{B_{\frac{d}{2}}(K)}} \phi_\varepsilon(x-y)\,dy = 0$ since

$\phi_\varepsilon(x-y) > 0$ iff $|x-y| < \varepsilon$, but when

$y \in \overline{B_{\frac{d}{2}}(K)}$ we have

$$\frac{d}{4} < \operatorname{dist}(x, \overline{B_{\frac{d}{2}}(K)}) = \operatorname{dist}(x, \overline{B_{\frac{d}{2}}(K)}) - \operatorname{dist}(y, \overline{B_{\frac{d}{2}}(K)})$$
$$\leq |x-y|.$$

$\therefore \operatorname{spt}(\rho) \subseteq \overline{B_{\frac{3d}{4}}(K)} \subset (a,b)$.

If $x \in K$, then $B_\varepsilon(x) \subset \overline{B_{\frac{d}{2}}(K)}$ and so

$$\rho(x) = \int_{\overline{B_{\frac{d}{2}}(K)}} \phi_\varepsilon(x-y)\,dy = \int_{\mathbb{R}} \phi_\varepsilon(x-y)\,dy = 1.$$

Finally, $0 \leq \rho(x) \leq \int_{\mathbb{R}} \phi_\varepsilon = 1$ for all x.

(b) Let $\psi(x) = \begin{cases} e^{\frac{1}{1-x^2}}, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$ from ①

and $\varphi(x) = \psi(x-1)$. Then $\varphi(x) = \psi(x-1)$ iff $x = \frac{1}{2}$ and

$$\max(\varphi,\psi) = \begin{cases} \varphi & x > \frac{1}{2} \\ \psi & x \leq \frac{1}{2} \end{cases}, \quad \min(\varphi,\psi) = \begin{cases} \psi & x > \frac{1}{2} \\ \varphi & x \leq \frac{1}{2} \end{cases}$$

Since $\varphi'(\frac{1}{2}) = \psi'(-\frac{1}{2}) = -\psi'(\frac{1}{2})$ and $\psi'(\frac{1}{2}) = \frac{16}{9} e^{\frac{4}{3}} \neq 0$ so $\max(\varphi,\psi), \min(\varphi,\psi)$ are not diff. at $x=\frac{1}{2}$, hence not in $\mathcal{D}(\mathbb{R})$.

(c) 1-dimensional case follows also from:

Take $K=\operatorname{spt}(u)$ so that $K\subset\Omega$ compact.
Let $\rho\in\mathcal{D}(\Omega)$, $0\leq\rho\leq 1$ and $\rho=1$ on K be the cut-off function constructed in Theorem 2.4 from lectures. Define

$$u_1=\rho\frac{(u+1)^2}{4}\ ,\quad u_2=\rho\frac{(u-1)^2}{4}\,.$$

Then $u_1,u_2\in\mathcal{D}(\Omega)$, $u_1,u_2\geq 0$ and $u_1-u_2=u$.

Hint in question was useless – students have been notified and given new 'hint':
'Note $4u=(u+1)^2-(u-1)^2$ and if v is cut-off function between $\operatorname{spt}(u)$ and $\partial\Omega$ then $vu=u$'.

---

(3) Put $\Omega_k=\{x\in\Omega\cap B_k(0): \operatorname{dist}(x,\partial\Omega)>\frac{1}{k}\}$ for $k\in\mathbb{N}$. Then by MCT

$$\int_\Omega |f-f\mathbb{1}_{\Omega_k}|^p\,dx=\int_{\Omega\setminus\Omega_k}|f|^p\,dx\xrightarrow[k\to\infty]{}0\,.$$

Take $k\in\mathbb{N}$ so $\left(\int_{\Omega\setminus\Omega_k}|f|^p\,dx\right)^{\frac{1}{p}}<\frac{\varepsilon}{2}$, put $O=\Omega_k$ and $h=f\mathbb{1}_O$. Then $h\in L^p(\mathbb{R}^n)$

and by Proposition 2.2 from lectures $\rho_\delta * h \in C^\infty(\Omega)$, $\|\rho_\delta * h - h\|_p \to 0$ as $\delta \to 0+$, where $(\rho_\delta)_{\delta>0}$ is the standard mollifier.

Since $h = 0$ a.e. on $\Omega \setminus O \subset \{x \in \Omega : dist(x, \partial\Omega) \leq \frac{1}{k}\}$ we have $\rho_\delta * h(x) = 0$ when $dist(x,\partial\Omega) < \frac{1}{k} - 2\delta$.

Take $\delta \in (0, \frac{1}{4k})$ so small that

$\|\rho_\delta * h - h\|_p < \frac{\varepsilon}{2}$.

Hereby $spt(\rho_\delta * h) \subseteq \{x \in \Omega \cap \overline{B_{k+2\delta}(0)} : dist(x, \partial\Omega) \geq \frac{1}{2k}\}$ $\subset \Omega$ so that $\rho_\delta * h$ has compact support in Ω. Thus $g = \rho_\delta * h \in \mathcal{D}(\Omega)$ and from Minkowski:

$$\|f - g\|_p \leq \|f - h\|_p + \|h - g\|_p < \varepsilon. \quad \square$$

---

(4) u_1 isn't a distribution as it's not of locally finite order:

Let $\rho \in \mathcal{D}(\mathbb{R})$ be a cut-off function satisfying $0 \leq \rho \leq 1$, $\rho = 1$ on $(-\frac{1}{2}, \frac{1}{2})$ and $spt(\rho) \subseteq (-1,1)$. Then $\rho(0) = 1$ but $\rho^{(k)}(0) = 0$ $\forall k \geq 1$.

If u_1 is a distribution, then there exist $c>0$, $m \in \mathbb{N}_0$ so

$$|\langle u_1,\varphi\rangle| = |\sum_{j=0}^{\infty} 2^{-j}\varphi^{(j)}(0)| \leq c\sum_{s=0}^{m} \sup_{[-1,1]}|\varphi^{(s)}|$$

For $n\in\mathbb{N}$ put $\varphi(x) = x^n \rho(x)$, $x\in\mathbb{R}$. Then $\varphi\in\mathcal{D}(-1,1)$ and by Leibniz:

$$\varphi^{(k)}(x) = \sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j}{\partial x^j}(x^n)\,\rho^{(k-j)}(x).$$

In particular $\varphi^{(k)}(0) = \begin{cases} n! & \text{if } k=n \\ 0 & \text{if } k\neq n\end{cases}$

and $$|\varphi^{(k)}(x)| \leq \sum_{j=0}^{k}\binom{k}{j} n(n-1)\cdots(n-j+1)|\rho^{(k-j)}(x)|$$

$$\leq 2^k n(n-1)\cdots(n-k+1)\max_{0\leq j\leq k}\sup|\rho^{(j)}(x)|,$$

hence

$$\text{RHS} \leq c\sum_{s=0}^{m}\sup_{[-1,1]}|\varphi^{(s)}| \leq c\sum_{s=0}^{m}2^s\frac{n!}{(n-s)!}\max_{0\leq j\leq s}\sup|\rho^{(j)}(x)|$$

$$\leq \underbrace{c\,2^{m+1}\max_{0\leq s\leq m}\sup|\rho^{(s)}|}_{c_m}\cdot\frac{n!}{(n-m)!}$$

$$\therefore\quad |\langle u_1,\varphi\rangle| = 2^{-n}n! \leq c_m\frac{n!}{(n-m)!}$$

and so $(n-m)! \leq c_m 2^n$.

If n is large this is false, and so u_1 isn't a distribution.

u_2 is a distribution on $\mathbb{R}$ (it's linear and sum is finite on each compact set ...)

u_3 isn't a distribution because it's not linear.

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(5) <u>Well-defined</u>: Let $\varphi \in \mathcal{D}(\mathbb{R})$.

Clearly the first two integrals are well-defined. For the integral over $[-a,a]$ put

$$\Phi(x) = \begin{cases} \dfrac{\varphi(x)-\varphi(0)}{|x|}, & x \neq 0 \\ \varphi'(0), & x = 0 \end{cases}.$$

Then Φ is piecewise continuous (cont. from right at 0); hence is (Riemann-) integrable on $[-a,a]$.

<u>Linearity</u> of T_a now follows from linearity of the integral.

Fix a compact set $K \subset \mathbb{R}$. Take $\ell > a$ so $K \subset [-\ell, \ell]$. For $\varphi \in \mathcal{D}(K)$:

$$\left| \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\varphi(x)}{|x|} \, dx \right| \leq 2 \int_a^\ell \frac{dx}{x} \sup|\varphi|$$

$$= 2 \log \frac{\ell}{a} \sup|\varphi| ,$$

$$\left| \int_{-a}^{a} \frac{\varphi(x) - \varphi(0)}{|x|} \, dx \right| \overset{FTC}{=} \left| \int_{-a}^{a} \int_0^1 \varphi'(tx) \, dt \, \frac{x}{|x|} \, dx \right|$$

$$\leq \int_{-a}^{a} \int_0^1 |\varphi'(tx)| \, dt \, dx$$

$$\leq 2a \sup|\varphi'|$$

$\therefore \; |\langle T_a, \varphi \rangle| \leq c \left(\sup|\varphi| + \sup|\varphi'| \right)$,

where $c = 2 \max(a, \log \frac{\ell}{a})$.

Hence $T_a \in \mathcal{D}'(\mathbb{R})$ (and is of order at most 1).

If $\varphi \in \mathcal{D}(\mathbb{R})$ with $\varphi(0) = 0$, then $\langle T_a, \varphi \rangle =$

$$\left(\int_{-\infty}^{-a} + \int_a^\infty \right) \frac{\varphi(x)}{|x|} \, dx + \int_{-a}^{a} \frac{\varphi(x) - 0}{|x|} \, dx = \int_{-\infty}^{\infty} \frac{\varphi(x)}{|x|} \, dx .$$

If $0 < a < b$, then for $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\langle T_a - T_b, \varphi \rangle = \left[\left(\int_{-\infty}^{-a} + \int_a^\infty \right) - \left(\int_{-\infty}^{-b} + \int_b^\infty \right) \right] \frac{\varphi(x)}{|x|} \, dx +$$

$$\left[\int_{-a}^{a} - \int_{-b}^{b}\right] \frac{\varphi(x)-\varphi(0)}{|x|}\,dx =$$

$$\left[\int_{-b}^{-a} + \int_{a}^{b}\right] \frac{\varphi(x)}{|x|}\,dx - \left[\int_{-b}^{-a} + \int_{a}^{b}\right] \frac{\varphi(x)-\varphi(0)}{|x|}\,dx =$$

$$\left[\int_{-b}^{-a} + \int_{a}^{b}\right] \frac{\varphi(0)}{|x|}\,dx = 2\int_a^b \frac{dx}{x}\,\varphi(0) = 2\log\frac{b}{a}\,\varphi(0)$$

$$\therefore \quad T_a - T_b = 2\log\frac{b}{a}\,\delta_0$$

6

If $v=0$, then $u=0$ and any constant c will do. Assume $v \neq 0$. Then we can find $\tilde{\chi} \in \mathcal{D}(\Omega)$ so $\langle v, \tilde{\chi}\rangle \neq 0$. Define $\chi = \dfrac{\tilde{\chi}}{\langle v, \tilde{\chi}\rangle}$. Then $\chi \in \mathcal{D}(\Omega)$ and $\langle v, \chi\rangle = 1$.

Now for $\varphi \in \mathcal{D}(\Omega)$, $\varphi - \langle v,\varphi\rangle\chi \in \mathcal{D}(\Omega)$ and $\langle v, \varphi - \langle v,\varphi\rangle \chi\rangle = 0$, hence also

$$0 = \langle u, \varphi - \langle v,\varphi\rangle\chi\rangle = \langle u,\varphi\rangle - \langle v,\varphi\rangle\langle u,\chi\rangle$$

and so $u = cv$ for $c = \langle u, \chi\rangle$. $\square$

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⑦ (optional)

By induction on $m \in \mathbb{N}_0$.

For $m=0$ we interpret E as the Heaviside function and so it's true. Assume it's true for some $m \in \mathbb{N}_0$: $\dfrac{d^{m+1}}{dx^{m+1}}\left[\dfrac{x_+^m}{m!}\right] = \delta_0$.

Then we have

$$\frac{d^{m+2}}{dx^{m+2}}\left[\frac{x_+^{m+1}}{(m+1)!}\right] = \frac{d^{m+1}}{dx^{m+1}}\left[\frac{d}{dx}\left(\frac{x_+^{m+1}}{(m+1)!}\right)\right].$$

For $\varphi \in \mathcal{D}(\mathbb{R})$: $\left\langle \frac{d}{dx}\left(\frac{x_+^{m+1}}{(m+1)!}\right), \varphi \right\rangle =$

$$\left\langle \frac{x_+^{m+1}}{(m+1)!}, -\varphi' \right\rangle = -\int_{-\infty}^{\infty} \frac{x_+^{m+1}}{(m+1)!}\varphi'(x)\,dx =$$

$$-\int_0^\infty \frac{x^{m+1}}{(m+1)!}\varphi'(x)\,dx \overset{\text{parts}}{=} -\left[\frac{x^{m+1}}{(m+1)!}\varphi(x)\right]_{x=0}^{x\to\infty} + \int_0^\infty \frac{x^m}{m!}\varphi(x)\,dx$$

$= \left\langle \frac{x_+^m}{m!}, \varphi \right\rangle$, and so we get from induction hypothesis

$$\frac{d^{m+2}}{dx^{m+2}}\left[\frac{x_+^{m+1}}{(m+1)!}\right] = \frac{d^{m}}{dx^{m}}\left[\frac{x_+^{m}}{m!}\right] = \delta_0. \ \square$$

8 (Optional)

(i) Let $\psi \in \mathcal{D}(\mathbb{R})$ be the function from 1. Put $\varphi(x) = c\psi(2x-1)$, $x \in \mathbb{R}$. Then clearly φ is C^∞ and since $\varphi(x) \neq 0$ iff $-1 < 2x-1 < 1$, ie, iff $0 < x < 1$, we have $\operatorname{spt}(\varphi) = [0,1]$. The constant $c > 0$ is chosen so $\int \varphi = 1$. Define

$$g(x) = \int_{-1}^{x} \big(\varphi(-t) - \varphi(t)\big)\,dt, \quad x \in \mathbb{R}.$$

By FTC, g is C^1 with $g'(x) = \varphi(-x) - \varphi(x)$, hence g is C^∞. Since $\varphi \equiv 0$ on $(-\infty, 0]$ and on $[1, \infty)$ we have $g(x) = 0$ for $x \leq -1$. For $x \geq 1$ we have $g(x) = \int_{-1}^{x} \varphi(-t)\,dt - \int^{1} \varphi(t)\,dt$ $= \int_{-1}^{0} \varphi(-t)\,dt - 1 = \int_0^1 \varphi - 1 = 0$, hence $\operatorname{spt}(g) \subseteq [-1,1]$. Also $g(0) = \int_{-1}^{0} \big(\varphi(-t) - \varphi(t)\big)\,dt$ $= \int_{-1}^{0} \varphi(-t)\,dt = \int_0^1 \varphi(t)\,dt = 1$, $g'(0) = \varphi(0) - \varphi(0)$ $= 0$ and $g^{(k+1)}(x) = (-1)^k \varphi^{(k)}(-x) - \varphi^{(k)}(x)$, so $g^{(k+1)}(0) = \big((-1)^k - 1\big)\varphi^{(k)}(0) = 0$ since $\operatorname{spt}(\varphi) = [0,1]$. (Can also be done using cut-off function from lectures.)

(ii) $g_n(x) = g\left(\frac{x}{\varepsilon_n}\right)\frac{a_n x^n}{n!}$, $x \in \mathbb{R}$, is C^∞ by Leibniz and $g_n(x) = 0$ when $\left|\frac{x}{\varepsilon_n}\right| \geq 1$, that is, when $|x| \geq \varepsilon_n$. Consequently, $g_n \in \mathcal{D}(\mathbb{R})$ and $\operatorname{spt}(g_n) \subseteq [-\varepsilon_n, \varepsilon_n]$.

$$g_0(0) = g(0)\frac{a_0 0^0}{0!} = a_0, \quad g_n(0) = 0 \quad \text{for } n \in \mathbb{N}.$$

By Leibniz we get for $k \in \mathbb{N}$:

$$g_n^{(k)}(x) = \sum_{j=0}^{k} \binom{k}{j} g^{(j)}\left(\frac{x}{\varepsilon_n}\right)\varepsilon_n^{-j}\frac{d^{k-j}}{dx^{k-j}}\left(\frac{a_n x^n}{n!}\right),$$

hence $g_n^{(k)}(0) \overset{(i)}{=} \left.\frac{d^k}{dx^k}\right|_{x=0}\left(\frac{a_n x^n}{n!}\right) = a_n \delta_{k,n}$.

Next, since $\operatorname{spt}(g) \subseteq [-1,1]$ we get for $0 \leq k < n$, $x \in \mathbb{R}$:

$$|g_n^{(k)}(x)| \leq \sum_{j=0}^{k}\binom{k}{j}\left|g^{(j)}\left(\frac{x}{\varepsilon_n}\right)\varepsilon_n^{-j} a_n \frac{x^{n-k+j}}{(n-k+j)!}\right|$$

$$\leq \sum_{j=0}^{k}\binom{k}{j}\sup|g^{(j)}|\,\varepsilon_n^{-j}\frac{|a_n|}{(n-k+j)!}\varepsilon_n^{n-k+j}$$

$$\leq \max_{0\leq j<n}\sup|g^{(j)}|\sum_{j=0}^{k}\binom{k}{j}\frac{|a_n|}{(n-k+j)!}\varepsilon_n^{n-k}$$

$$\leq \max_{0\leq j<n}\sup|g^{(j)}|\sum_{j=0}^{k}\binom{k}{j}\frac{|a_n|}{n!}\varepsilon_n^{n-k}$$

$$= \frac{|a_n|}{n!} 2^k \max_{0 \leq j < n} \sup |g^{(j)}| \, \varepsilon_n^{n-k}$$

$$\leq \frac{|a_n| 2^n}{n!} \max_{0 \leq j < n} \sup |g^{(j)}| \cdot \varepsilon_n \quad \text{provided } \varepsilon_n \leq 1.$$

Take $\varepsilon_n = \dfrac{2^{-n}}{1 + \dfrac{|a_n| 2^n}{n!} \max\limits_{0 \leq j < n} \sup |g^{(j)}|}$ to

conclude. □

(iii) With above ε_n, put $f(x) = \sum_{n=0}^{\infty} g_n(x)$, $x \in \mathbb{R}$.

Because $|g_n(x)| \leq 2^{-n}$ for all x it follows from Weierstrass' M-test that the series is uniformly convergent on $\mathbb{R}$. As each g_n is C^∞, f is in particular continuous.

Assume for some $k \in \mathbb{N}_0$, f is C^k with $f^{(k)}(x) = \sum_{n=0}^{\infty} g_n^{(k)}(x)$, $x \in \mathbb{R}$.

Then by (ii) (choice of ε_n): $|g_n^{(k+1)}(x)| \leq 2^{-n}$ for $x \in \mathbb{R}$, $n > k+1$, so Weierstrass again gives uniform convergence of $\sum_{n > k+1} g_n^{(k+1)}(x)$ on $\mathbb{R}$, and hence of $\sum_{n=0}^{\infty} g_n^{(k+1)}(x)$ on $\mathbb{R}$.

By a result from prelims f is C^{k+1} with $f^{(k+1)}(x) = \sum_{n=0}^{\infty} g_n^{(k+1)}(x)$ for $x \in \mathbb{R}$.

Thus by induction, f is C^∞. Since g_n is supported in $[-\varepsilon_n, \varepsilon_n]$ and $\varepsilon_1 \leq 1$ it follows that f is supported in $[-1,1]$.

Finally, $f^{(n)}(0) = \sum_{k=0}^{\infty} g_k^{(n)}(0) = a_n$ for all $n \in \mathbb{N}_0$.

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9 (Optional)

(i) $(h_r * h_s)(x) = \frac{1}{rs} \int_{-\infty}^{\infty} \mathbb{1}_{(0,r)}(x-y)\, \mathbb{1}_{(0,s)}(y)\, dy =$

$\frac{1}{rs} \int_0^s \mathbb{1}_{(0,r)}(x-y)\, dy = \frac{1}{rs} \int_{x-s}^{x} \mathbb{1}_{(0,r)}(t)\, dt =$

$$\frac{1}{rs} \mathcal{L}^1((x-s,x) \cap (0,r)) = \begin{cases} 0 & x \leq 0 \\ x/rs & 0 < x \leq r \\ 1/s & r < x \leq s \\ \frac{r+s-x}{rs} & s < x \leq r+s \\ 0 & r+s < x \end{cases}$$

so continuous by inspection (in fact, it's Lipschitz continuous with Lipschitz constant $\frac{1}{rs}$)

We also note that $\mathrm{spt}(h_r * h_s) = [0, r+s]$, and that $0 \leq h_r * h_s \leq \frac{1}{s}$.

(ii) For $x \in \mathbb{R}$, $(h_r * u)(x) = \frac{1}{r}\int_{-\infty}^{\infty} \mathbb{1}_{(0,r)}(x-y)u(y)dy$

$= \frac{1}{r}\int_{x-r}^{x} u(y)\,dy$, and so $h_r * u$ is C^1 by

FTC with $(h_r * u)'(x) = \frac{1}{r}(u(x) - u(x-r))$.

But since u is C^k, $h_r * u$ must then be

C^{k+1} with $(h_r * u)^{(k+1)}(x) = \frac{1}{r}(u^{(k)}(x) - u^{(k)}(x-r))$.

By inspection we get from $\mathrm{spt}(u) \subseteq [a,b]$

that $\mathrm{spt}(h_r * u) \subseteq [a, b+r]$.

(iii) We have $u_1 = h_{r_0} * h_{r_1}$ is C^0 with

$\mathrm{spt}(u_1) \subseteq [0, r_0 + r_1]$ and $0 \leq u_1 \leq \frac{1}{r_0}$, so

claim is true for $n=1$. Now assume it's

true for $m \leq n \in \mathbb{N}$, any $0 < r_0 \leq r_1 \leq \dots \leq r_m$

$v_m = ~~u_m~~$ $h_{r_0} * h_{r_1} * \dots * h_{r_m}$ is C_c^{m-1}

with support in $[0, r_0 + \dots + r_m]$ and $v_m^{(k)}$ has

values in $[-\frac{2^k}{r_0 \cdots r_k}, \frac{2^k}{r_0 \cdots r_k}]$, $0 \leq k < m$.

Consider $u_{n+1} = h_{r_0} * \dots * h_{r_{n+1}} = h_{r_0} * v_n$,

where $v_n = h_{r_1} * \dots * h_{r_{n+1}}$. By induction

hypothesis, v_n is C_c^{n-1} with

$\operatorname{spt}(v_n) \subseteq [0, r_1 + \cdots + r_{n+1}]$ and

$$|v_n^{(k)}| \leq \frac{2^k}{r_1 r_2 \cdots r_{k+1}} \quad \text{for } 0 \leq k < n.$$

Now by (ii) we get $u_{n+1} = h_{r_0} * v_n$ is C_c^k with $\operatorname{spt}(u_{n+1}) \subseteq [0, r_0 + \cdots + r_{n+1}]$ and

$$u_{n+1}^{(k)}(x) = \frac{v_n^{(k-1)}(x) - v_n^{(k-1)}(x - r_0)}{r_0}.$$

Hence

$$|u_{n+1}^{(k)}(x)| \leq \frac{1}{r_0}\left(|v_n^{(k-1)}(x)| + |v_n^{(k-1)}(x - r_0)|\right)$$

$$\leq \frac{1}{r_0}\left(\frac{2^{k-1}}{r_1 \cdots r_k} + \frac{2^{k-1}}{r_1 \cdots r_k}\right)$$

$$= \frac{2^k}{r_0 r_1 \cdots r_k} \quad \text{for } k < n+1,$$

and the claim follows by induction.

(iv) For $m, n \in \mathbb{N}$ and $x \in \mathbb{R}$:

$$|u_{m+n}(x) - u_n(x)| = \left|\int_{-\infty}^{\infty} (u_n(x-y) - u_n(x))\, v(y)\, dy\right|$$

where $v(y) = (h_{r_{n+1}} * \cdots * h_{r_{n+m}})(y)$. Recall that $\int_{-\infty}^{\infty} v = 1$, and $\operatorname{spt}(v) \subseteq [0, r_{n+1} + \cdots + r_{n+m}]$ so using FTC and the bound from (iii)

$$|u_n(x-y)-u_n(x)| = \left|\int_0^1 u_n'(x-ty)\,y\,dt\right|$$

$$\leq \frac{2}{r_0 r_1}|y|,$$

we get $|u_{m+n}(x) - u_n(x)| \leq \int_0^{r_{n+1}+\dots+r_{n+m}} \frac{2}{r_0 r_1}|y|\,v(y)\,dy$

$\leq \frac{2}{r_0 r_1}(r_{n+1}+\dots+r_{n+m})$. It follows that (u_n) is uniformly Cauchy on $\mathbb{R}$, and so $u(x) = \lim_{n\to\infty} u_n(x)$ exists as a continuous function. Since with $v_n = h_{r_1} * \dots * h_{r_n}$ and $u_n = h_{r_0} * v_n$ we have as in (iii)

$$u_n^{(k)}(x) = \frac{v_n^{(k-1)}(x) - v_n^{(k-1)}(x-r_0)}{r_0}, \quad k<n,$$

we can repeat the above argument to see that $(u_n^{(k)})$ is a uniform Cauchy sequence on $\mathbb{R}$, and hence that u is C^k. From (iii) we then deduce that $\operatorname{spt}(u) \subseteq [0,R]$ and

$$|u^{(k)}| \leq \frac{2^k}{r_0 r_1 \cdots r_k}, \quad k \in \mathbb{N}_0. \quad \square$$