① Let $\varphi \in \mathcal{D}(a,b)$ and consider

$\langle g_j, \varphi \rangle = \int_{-\infty}^{\infty} \varphi(x)\, g(jx)\, dx$, where we extend φ to $\mathbb{R}\setminus(a,b)$ by defining $\varphi = 0$ off (a,b).

Take $\alpha, \beta \in \mathbb{Z}$ so $\alpha T \leq a$, $\beta T \geq b$. Assume first that $g = \mathbb{1}_{(c,d]}$ on $(0,T]$. Then

$$\langle g_j, \varphi \rangle = \sum_{k=0}^{j(\beta-\alpha)} \int_{\alpha T + k\frac{T}{j} + \frac{c}{j}}^{\alpha T + k\frac{T}{j} + \frac{d}{j}} \varphi(x)\, dx =$$

$$\sum_{k=0}^{j(\beta-\alpha)} \left\{ \int_{\alpha T + k\frac{T}{j} + \frac{c}{j}}^{\alpha T + k\frac{T}{j} + \frac{d}{j}} \left(\varphi(x) - \varphi\left(\alpha T + k\frac{T}{j}\right) \right) dx + \varphi\left(\alpha T + k\frac{T}{j}\right) \frac{d-c}{j} \right\} =$$

$I + II$, say. We estimate I using the Lipschitz bound $|\varphi(x) - \varphi(y)| \leq \max|\varphi'|\, |x-y|$:

$$|I| \leq \sum_{k=0}^{j(\beta-\alpha)} \max|\varphi'| \left(\frac{T}{j}\right)^2 = \max|\varphi'| \left(\frac{T}{j}\right)^2 \left(j(\beta-\alpha)+1\right)$$

$\xrightarrow[j]{} 0$. Next, $II = \sum_{k=0}^{j(\beta-\alpha)} \varphi\left(\alpha T + k\frac{T}{j}\right) \frac{d-c}{j} =$

$$\frac{d-c}{T} \sum_{k=0}^{j(\beta-\alpha)} \varphi\left(\alpha T + k\frac{T}{j}\right) \frac{T}{j} = \frac{d-c}{T} \sum_{k=0}^{j(\beta-\alpha)} \left\{ \int_{\alpha T + k\frac{T}{j}}^{\alpha T + (k+1)\frac{T}{j}} \varphi(x)\, dx \right.$$

$$\left. + \int_{\alpha T + k\frac{T}{j}}^{\alpha T + (k+1)\frac{T}{j}} \left(\varphi\left(\alpha T + k\frac{T}{j}\right) - \varphi(x) \right) dx \right\} = II' + II'', \text{ say.}$$

Clearly, $II' = \frac{d-c}{T} \int_{\alpha T}^{\beta T} \varphi(x)\, dx$ and we use the Lipschitz estimate again to estimate $\underset{j(\beta-\alpha)}{\mathrm{II}''}$:

$$|\mathrm{II}''| \leq \frac{d-c}{T} \sum_{k=0}^{j(\beta-\alpha)} \max|\varphi'| \left(\frac{T}{j}\right)^2 \xrightarrow[j]{} 0 .$$

Thus $\langle g_j, \varphi \rangle \xrightarrow[j]{} \frac{1}{T} \int_0^T g\,dx \int_a^b \varphi\,dx$ in this case. By Linearity of the integral and algebra of limits this remains true for T periodic step functions.

Let $g \in L^1_{loc}(\mathbb{R})$ be T periodic.

Fix $\varphi \in \mathcal{D}(a,b)$ and $\varepsilon > 0$. For an $m > 0$ we select $s \in L^{step}(0,T]$: so $\|g - s\|_1 < \frac{\varepsilon}{m}$.

Below we'll choose m appropriately. Extend s to $\mathbb{R}$ by T periodicity and observe that with $s_j(x) = s(jx)$ we have

$$\langle s_j, \varphi \rangle \xrightarrow[j]{} \frac{1}{T} \int_0^T s\,dx \int_a^b \varphi\,dx .$$

Now $\left| \langle g_j, \varphi \rangle - \frac{1}{T} \int_0^T g\,dx \int_a^b \varphi\,dx \right| \leq$

$$|\langle g_j - s_j, \varphi \rangle| + \left| \langle s_j, \varphi \rangle - \frac{1}{T} \int_0^T s\,dx \int_a^b \varphi\,dx \right| +$$

$$\left| \frac{1}{T} \int_0^T (s-g)\,dx \int_a^b \varphi\,dx \right| = \mathrm{I} + \mathrm{II} + \mathrm{III} , \text{ say.}$$

Clearly, $\mathrm{II} \xrightarrow[j]{} 0$ for m, ε fixed.

$$|\mathrm{I}| \leq \int_a^b |g(jx) - s(jx)||\varphi(x)|\,dx \leq$$

$$\|\varphi\|_\infty \frac{1}{j} \int_{ja}^{jb} |g(y) - s(y)|\,dy \leq \|\varphi\|_\infty \frac{1}{j} \int_{j\alpha T}^{j\beta T} |g(y)-s(y)|\,dy$$

$$= \|\varphi\|_\infty \frac{j(\beta-\alpha)}{j} \int_0^T |g-s|\,dy \leq \|\varphi\|_\infty (\beta-\alpha)\frac{\varepsilon}{m},$$

$$|\mathrm{III}| \leq \frac{1}{T}\int_0^T |s-g|\,dx\,\|\varphi\|_1 \leq \frac{\|\varphi\|_1}{T}\cdot\frac{\varepsilon}{m}$$

Select $m>0$ so $\dfrac{\|\varphi\|_\infty(\beta-\alpha)}{m} + \dfrac{\|\varphi\|_1}{mT} < 1$,

whereby $|\mathrm{I}| + |\mathrm{III}| < \varepsilon$ uniformly in j,

hence

$$\limsup_{j\to\infty} \left| \langle g_j, \varphi\rangle - \frac{1}{T}\int_0^T g\,dx \int_a^b \varphi\,dx \right| < \varepsilon.$$

---

(2) u is piecewise C^1 and thus L^1_{loc}, hence we may consider $u \in \mathcal{D}'(\mathbb{R})$. Now for $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\langle u', \varphi\rangle = \langle u, -\varphi'\rangle = -\int_{-\infty}^{\infty} u(x)\varphi'(x)\,dx =$$

$$-\int_{-\infty}^{0} f(x)\varphi'(x)\,dx - \int_0^\infty g(x)\varphi'(x)\,dx \overset{\text{parts}}{=}$$

$$-\left[f(x)\varphi(x)\right]_{x\to-\infty}^{x=0} + \int_{-\infty}^0 f'(x)\varphi(x)\,dx - \left[g(x)\varphi(x)\right]_{x=0}^{x\to\infty} + \int_0^\infty g'(x)\varphi(x)\,dx$$

$$-f(0)\varphi(0) + \int_{-\infty}^{0} f'\varphi\,dx + g(0)\varphi(0) + \int_0^{\infty} g'\varphi\,dx =$$

$$\langle (g(0)-f(0))\delta_0, \varphi\rangle + \int_{\mathbb{R}} \left(f'\mathbb{1}_{(-\infty,0)} + g'\mathbb{1}_{(0,\infty)}\right)\varphi\,dx$$

so $$u' = (g(0)-f(0))\delta_0 + f'\mathbb{1}_{(-\infty,0)} + g'\mathbb{1}_{(0,\infty)}.$$

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3) (a) $u_\alpha(x) = |x|^\alpha$, $\alpha > -n$.

u_α is clearly locally integrable in $\mathbb{R}^n \setminus \{0\}$.

Integrating in polar coordinates we get

$$\int_{B_1(0)} |u_\alpha(x)|\,dx = \int_0^1 \int_{|x|=r} |x|^\alpha\,dS_x\,dr \overset{(*)}{=}$$

$$\int_0^1 r^\alpha \cdot \omega_n r^{n-1}\,dr =$$

$$\left[\frac{\omega_n}{n+\alpha} r^{n+\alpha}\right]_{r\to 0+}^{r=1} = \frac{\omega_n}{n+\alpha} < \infty$$

(*) ω_n = surface area of $\partial B_1(0)$ in $\mathbb{R}^n$ = $\dfrac{\mathcal{L}^n(B_1(0))}{n}$.

since $\alpha > -n$.

Thus $u_\alpha \in L^1_{loc}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

(b) For $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} d_r\varphi\,\psi\,dx = \int_{\mathbb{R}^n} \varphi(rx)\psi(x)\,dx \overset{\substack{y=rx\\ dy=r^n dx}}{=}$$

$$\int_{\mathbb{R}^n} \varphi(y)\psi\left(\frac{y}{r}\right)\frac{1}{r^n}\,dy = \int_{\mathbb{R}^n} \varphi\, r^{-n} d_{r^{-1}}\psi\,dy$$

an adjoint identity with $r^{-n} d_{\frac{1}{r}} : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ clearly linear and continuous. We may thus define for $u \in \mathcal{D}'(\mathbb{R}^n)$:

$$\langle d_r u, \varphi \rangle \stackrel{\text{def}}{=} \langle u, \tfrac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

(c) For $\varphi \in \mathcal{D}(\mathbb{R}^n)$: $\langle d_r u_\alpha, \varphi \rangle =$

$$\langle u_\alpha, \tfrac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle = \int_{\mathbb{R}^n} |x|^\alpha \tfrac{1}{r^n} \varphi(\tfrac{x}{r})\, dx \overset{y = \frac{x}{r}}{\underset{dy = \frac{1}{r^n} dx}{=}}$$

$$\int_{\mathbb{R}^n} |ry|^\alpha \tfrac{1}{r^n} \varphi(y) \cdot r^n dy = r^\alpha \int_{\mathbb{R}^n} |y|^\alpha \varphi(y)\, dy =$$

$r^\alpha \langle u_\alpha, \varphi \rangle$, as required.

(d) $\langle d_r \delta_0, \varphi \rangle = \langle \delta_0, \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle = \frac{1}{r^n} \varphi(\frac{0}{r}) =$

$\frac{1}{r^n} \langle \delta_0, \varphi \rangle$, that is, $d_r \delta_0 = \frac{1}{r^n} \delta_0$ for all $r > 0$.

(e) $\langle d_r (x_j u), \varphi \rangle = \langle x_j u, \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle =$

$\langle u, x_j \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle$.

Now $x_j \left(\frac{1}{r^n} d_{\frac{1}{r}} \varphi \right)(x) = x_j \frac{1}{r^n} \varphi(\frac{x}{r}) = \frac{1}{r^{n-1}} (x_j \varphi)(\frac{x}{r}) =$

$r \frac{1}{r^n} d_{\frac{1}{r}} (x_j \varphi)$ so $\langle u, x_j \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle = r \langle u, \frac{1}{r^n} d_{\frac{1}{r}} (x_j \varphi) \rangle$

$= r \langle d_r u, x_j \varphi \rangle = r \cdot r^\beta \langle u, x_j \varphi \rangle \overset{BH}{=} r \langle x_j u, \varphi \rangle$

Hence $d_r(x_j u) = r^{\beta+1} u$, as required.

$< d_r(D_j u), \varphi > = < D_j u, \frac{1}{r^n} d_{\frac{1}{r}} \varphi > =$
$- < u, D_j \frac{1}{r^n} d_{\frac{1}{r}} \varphi >$. Calculate : $D_j(\frac{1}{r^n} d_{\frac{1}{r}} \varphi)(x) =$
$\frac{1}{r^n} D_j(\varphi(\frac{x}{r})) = \frac{1}{r^{n+1}} (D_j \varphi)(\frac{x}{r}) = \frac{1}{r} \cdot \frac{1}{r^n} d_{\frac{1}{r}} (D_j \varphi)(x)$,

hence $< u, D_j \frac{1}{r^n} d_{\frac{1}{r}} \varphi > = \frac{1}{r} < d_r u, D_j \varphi >$ and

so $< d_r(D_j u), \varphi > = -\frac{1}{r} < d_r u, D_j \varphi > =$
$-\frac{1}{r} r^\beta < u, D_j \varphi > = r^{\beta-1} < D_j u, \varphi >$. Thus
$d_r(D_j u) = r^{\beta-1} D_j u$, as required.

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we calculate :

$$< \sum_{j=1}^n x_j D_j u, \varphi > = < u, -\sum_{j=1}^n D_j(x_j \varphi) > =$$

$$< u, -\sum_{j=1}^n (\varphi + x_j D_j \varphi) > = < u, -n\varphi - x \cdot \nabla \varphi >.$$

By above, $\sum_{j=1}^n x_j D_j u$ is homogeneous of degree β, as u is: $d_r u = r^\beta u$ for $r > 0$,

or :
$$< u, r^{-n} d_{r^{-1}} \varphi > = r^\beta < u, \varphi > \text{ for } r > 0.$$

Consequently, $\beta < u, \varphi > = \frac{d}{dr}\Big|_{r=1} < u, r^{-n} d_{r^{-1}} \varphi >$

Note

$$\frac{d}{dr}\Big|_{r=1}\left(r^{-n}d_{r^{-1}}\varphi\right)(x) = -n\varphi(x) - \nabla\varphi(x).x,$$

and for $0<\varepsilon<1$, $0<|t|<\varepsilon$:

$$\Delta_t(x) = \frac{1}{t}\left(\left(1+t\right)^{-n}\varphi\left(\frac{x}{1+t}\right) - \varphi(x)\right) \xrightarrow[t\to 0]{} -n\varphi(x) - D\varphi(x).x$$

uniformly in $x\in\mathbb{R}^n$. If we let

$$R = \sup\left\{\frac{|x|}{1-\varepsilon} : x\in \operatorname{spt}(\varphi)\right\},$$

then $\operatorname{spt}\Delta_t \subset B_R(0)$ for $0<|t|<\varepsilon$, and for $\alpha\in\mathbb{N}_0^n$ we have

$$D^\alpha\Delta_t(x) = \frac{1}{t}\left(\left(1+t\right)^{-n-|\alpha|}(D^\alpha\varphi)\left(\frac{x}{1+t}\right) - D^\alpha\varphi(x)\right)$$

$$\xrightarrow[t\to 0]{} -(n+|\alpha|)D^\alpha\varphi(x) - \nabla(D^\alpha\varphi)(x).x$$

uniformly in $x\in\mathbb{R}^n$. Observe that

$$D^\alpha\left(-n\varphi(x) - \nabla\varphi(x).x\right) = -nD^\alpha\varphi(x) - \sum_{j=1}^{n} D^\alpha(x_jD_j\varphi(x))$$

and by Leibniz

$$D^\alpha(x_jD_j\varphi(x)) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}D^\beta(x_j)\,D^{\alpha-\beta}(D_j\varphi(x)).$$

Since

$$D^\beta(x_j) = \begin{cases} x_j & \text{if } \beta=0\\ 1 & \text{if } \beta=e_j\\ 0 & \text{else}\end{cases}$$

the above simplifies to

$x_j D^{\alpha+e_j}\varphi(x) + \binom{\alpha}{e_j} D^{\alpha-e_j}(D_j\varphi(x)) =$

$x_j D_j(D^\alpha\varphi)(x) + \alpha_j D^\alpha\varphi(x)$, and consequently

$$D^\alpha(-n\varphi(x) - \nabla\varphi(x).x) = -n D^\alpha\varphi(x) - \sum_{j=1}^n (x_j D_j(D^\alpha\varphi)(x) + \alpha_j D^\alpha\varphi(x)) = -(n+|\alpha|)D^\alpha\varphi(x) - x.\nabla(D^\alpha\varphi)(x).$$

We have shown that $\Delta_t(x) \xrightarrow[t\to 0]{} -n\varphi(x) - \nabla\varphi(x).x$ in $\mathcal{D}(\mathbb{R}^n)$, and hence

$$\beta\langle u,\varphi\rangle = \frac{d}{dr}\Big|_{r=1} \langle u, r^{-n} d_{r^{-1}}\varphi\rangle = \lim_{t\to 0}\langle u,\Delta_t\rangle = \langle u, \frac{d}{dr}\Big|_{r=1} r^{-n} d_{r^{-1}}\varphi\rangle =$$

$$\langle u, -n\varphi - \nabla\varphi.x\rangle = \langle \sum_{j=1}^n x_j D_j u, \varphi\rangle,$$

as required.

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(4) See Section 2.5 in Strichartz: A Guide to Distribution Theory and Fourier Transforms, World Scientific.

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5) For $\varphi \in \mathcal{D}(\mathbb{R})$:

$\langle x\delta_0, \varphi \rangle = \langle \delta_0, x\varphi \rangle = 0 \cdot \varphi(0) = 0$ so

$x\delta_0 = 0$

Next, for $\varphi \in \mathcal{D}(\mathbb{R})$ we have by FTC for $x \in \mathbb{R}$ since $\eta(0) = 1$:

$$\varphi(x) - \varphi(0)\eta(x) = \int_0^1 \frac{d}{dt}\left(\varphi(tx) - \varphi(0)\eta(tx)\right) dt$$

$$= \int_0^1 \left(\varphi'(tx) - \varphi(0)\eta'(tx)\right) dt \cdot x.$$

Put $\psi(x) = \int_0^1 \left(\varphi'(tx) - \varphi(0)\eta'(tx)\right) dt$, $x \in \mathbb{R}$. Then ψ is C^∞ and supported in $[-R, R]$, where $R = \sup\{|x| : x \in \mathrm{spt}(\varphi) \cup \mathrm{spt}(\eta)\}$.

The latter follows from $\psi(x) = \dfrac{\varphi(x) - \varphi(0)\eta(x)}{x} = 0$ for $|x| \geq R$. Thus $\psi \in \mathcal{D}(\mathbb{R})$ and we have $\varphi = \varphi(0)\eta + x\psi$.

Assume $xu = 0$. Then $\langle u, \varphi \rangle = \langle u, \varphi(0)\eta + x\psi \rangle = \langle u, \eta \rangle \varphi(0) + \langle xu, \psi \rangle = \langle \langle u, \eta \rangle \delta_0, \varphi \rangle$. Since clearly $c\delta_0$ $(c \in \mathbb{R})$ is a solution we have found GS: $u = c\delta_0$, $c \in \mathbb{R}$.

(i) If $u \in \mathcal{D}(\mathbb{R})$ we have for $\varphi \in \mathcal{D}(\mathbb{R})$ by Fubini: $\langle \theta * u, \varphi \rangle = \iint \theta(x-y)u(y)\,dy\,\varphi(x)\,dx =$

$\iint \theta(x-y)\varphi(x)\,dx\,u(y)\,dy = \langle u, \tilde{\theta} * \varphi \rangle$

Clearly, $\varphi \mapsto \tilde{\theta} * \varphi$ is a linear map of $\mathcal{D}(\mathbb{R})$ and since $\frac{d^m}{dx^m} \tilde{\theta} * \varphi = \tilde{\theta} * \varphi^{(m)}$ it's also continuous. The adjoint identity scheme then allow us to define $\theta * u \in \mathcal{D}'(\mathbb{R})$ for each $u \in \mathcal{D}'(\mathbb{R})$ by the rule

$$\langle \theta * u, \varphi \rangle \overset{\text{def}}{=} \langle u, \tilde{\theta} * \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

(ii) For each $x \in \mathbb{R}$ we put $\theta^x(y) = \theta(x-y)$, $y \in \mathbb{R}$, so that $\theta^x \in \mathcal{D}(\mathbb{R})$ and $\langle u, \theta^x \rangle$ is well-defined. We claim that $x \mapsto \langle u, \theta^x \rangle$ is C^∞ and $\langle \theta * u, \varphi \rangle = \int \langle u, \theta^x \rangle \varphi(x)\,dx$ for $\varphi \in \mathcal{D}(\mathbb{R})$.

Note that for $h \neq 0$,

$$\frac{\theta^{x+h}(y) - \theta^x(y)}{h} = \frac{\theta(x+h-y) - \theta(x-y)}{h}$$

$$= \int_0^1 \theta'(th + x - y)\,dt$$

and for any $m \in \mathbb{N}_0$:

$$\frac{d^m}{dy^m}\left(\frac{\theta^{x+h}(y)-\theta^x(y)}{h}\right) = (-1)^m\int_0^1 \theta^{(m+1)}(th+x-y)\,dt.$$

Since $\frac{1}{h}(\theta^{x+h}-\theta^x)$ is supported in $[-R,R]$ for $0<|h|<1$, where $R=\sup\{|x|+1+|t| : t\in \mathrm{spt}(\theta)\}$ we see that $\dfrac{\theta^{x+h}-\theta^x}{h} \xrightarrow[h\to 0]{} \theta'(x-y)$ in $\mathcal{D}(\mathbb{R})$

and consequently that

$$\frac{1}{h}\left(\langle u,\theta^{x+h}\rangle - \langle u,\theta^x\rangle\right) = \left\langle u, \frac{\theta^{x+h}-\theta^x}{h}\right\rangle \xrightarrow[h\to 0]{} \langle u, \theta'(x-\cdot)\rangle$$

The function $x\mapsto \langle u,\theta^x\rangle$ is therefore differentiable. Because the above argument applies equally well to $\theta^{(m)}(x-\cdot)$ we infer by induction on m that $x\mapsto\langle u,\theta^x\rangle$ is C^∞. Now by linearity of u we get for $\varphi\in\mathcal{D}(\mathbb{R})$:

$$\int_{-\infty}^{\infty}\langle u,\theta^x\rangle\varphi(x)\,dx = \int_{-\infty}^{\infty}\langle u,\varphi(x)\theta^x\rangle\,dx.$$

Take $a,b\in\mathbb{R}$ so $\mathrm{spt}(\varphi)\subset(a,b)$ and let $a=x_0<x_1<\cdots<x_n=b$ be a partition with $x_j-x_{j-1}=\frac{b-a}{n}$. By uniform continuity of $(x,y)\mapsto\varphi(x)\theta^x(y)$ we have

$$\int_{-\infty}^{\infty} \varphi(x)\,\theta^{x}(y)\,dx = \lim_{n\to\infty} \sum_{j=0}^{n-1} \varphi(x_j)\,\theta^{x_j}(y)\,(x_{j+1}-x_j)$$

uniformly in $y \in \mathbb{R}$. Put $R = \sup\{|a|+|b|+|c| ; \operatorname{spt}\theta\}$ and note that $y \mapsto \sum_{j=0}^{n-1} \varphi(x_j)\theta^{x_j}(y)(x_{j+1}-x_j)$ is supported in $[-R,R]$, is C^∞ smooth and for each $m \in \mathbb{N}$,

$$\int_{-\infty}^{\infty} \varphi(x)\frac{d^m}{dy^m}\theta^{x}(y)\,dx = \lim_{n\to\infty}\sum_{j=0}^{n-1} \varphi(x_j)\frac{d^m\theta^{x_j}}{dy^m}(y)(x_{j+1}-x_j)$$

uniformly in $y \in \mathbb{R}$. The convergence is therefore in the $\mathcal{D}(\mathbb{R})$ sense, so by continuity of $u \in \mathcal{D}'(\mathbb{R})$ we get

$$\left\langle u, \sum_{j=0}^{n-1} \varphi(x_j)\theta^{x_j}(\cdot)(x_{j+1}-x_j)\right\rangle \xrightarrow[n]{} \left\langle u, \int_{-\infty}^{\infty} \varphi(x)\theta^{x}(\cdot)\,dx\right\rangle$$

$$= \langle u, \tilde{\theta} * \varphi\rangle .$$

Now by linearity LHS $= \sum_{j=0}^{n-1} \varphi(x_j)\langle u, \theta^{x_j}\rangle (x_{j+1}-x_j)$ and since $x \mapsto \varphi(x)\langle u, \theta^x\rangle \in \mathcal{D}(a,b)$ it follows that LHS $\xrightarrow[n]{} \int_{-\infty}^{\infty} \varphi(x)\langle u, \theta^x\rangle\,dx$, as required. We may therefore define

$(\theta * u)(x) \overset{\text{def}}{=} \langle u, \theta^x\rangle$, $x \in \mathbb{R}$. $\square$

(iii) From (ii) we have $\rho_\varepsilon * u \in C^\infty(\mathbb{R})$ and $\langle \rho_\varepsilon * u, \varphi \rangle = \langle u, \rho_\varepsilon * \varphi \rangle$ since $\tilde{\rho} = \rho$.

Now $\rho_\varepsilon * \varphi \xrightarrow[\varepsilon \to 0+]{} \varphi$ in $\mathcal{D}(\mathbb{R})$ since $\operatorname{spt}(\rho_\varepsilon * \varphi) \subseteq \overline{B_\varepsilon(\operatorname{spt}(\varphi))}$ and $(\rho_\varepsilon * \varphi)^{(m)} = \rho_\varepsilon * \varphi^{(m)}$, so by continuity of u we have

$$\langle \rho_\varepsilon * u, \varphi \rangle \xrightarrow[\varepsilon \to 0+]{} \langle u, \varphi \rangle . \quad \square$$

(iv) For $k \in \mathbb{N}$ put $\phi_k(x) = (\rho * 1\!\!1_{(-k,k)})(x)$.

Then $\phi_k \in \mathcal{D}(\mathbb{R})$, $0 \leq \phi_k \leq 1$, $\phi_k = 0$ for $|x| > k+1$ and $\phi_k = 1$ for $|x| \leq k-1$. Now $\rho_\varepsilon * (\phi_k u) \in \mathcal{D}'(\mathbb{R})$ and for $\varphi \in \mathcal{D}(\mathbb{R})$

$$\lim_{k \to \infty} \lim_{\varepsilon \to 0+} \langle \rho_\varepsilon * (\phi_k u), \varphi \rangle = \langle u, \varphi \rangle .$$

Put $u_j = \rho_{\varepsilon_j} * (\phi_{k_j} u)$, where (ε_j), (k_j) are any sequences so $\varepsilon_j \downarrow 0$ and $k_j \uparrow \infty$.

We have $u_j \in \mathcal{D}(\mathbb{R})$ and for $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\langle u_j, \varphi \rangle = \langle u, \phi_{k_j} \rho_{\varepsilon_j} * \varphi \rangle .$$

Here $\phi_{k_j} \rho_{\varepsilon_j} * \varphi \xrightarrow[j]{} \varphi$ in $\mathcal{D}(\mathbb{R})$:

$\operatorname{spt}(\phi_{k_j} \rho_{\varepsilon_j} * \varphi) \subseteq \overline{B_{\varepsilon_j}(\operatorname{spt}(\varphi))}$ and for

$m \in \mathbb{N}_0$ we have by Leibniz:

$$\frac{d^m}{dx^m}(\psi_{k_j} \cdot \rho_{\varepsilon_j} * \varphi) = \sum_{i=0}^{m} \binom{m}{i} \psi_{k_j}^{(i)} \, \rho_{\varepsilon_j} * \varphi^{(m-i)}$$

Since $\psi_{k_j} \equiv 1$ on $\operatorname{spt}(\rho_{\varepsilon_j} * \varphi)$ for large j we have for such j,

$$\frac{d^m}{dx^m}(\psi_{k_j} \, \rho_{\varepsilon_j} * \varphi) = \rho_{\varepsilon_j} * \varphi^{(m)} \xrightarrow[j]{} \varphi^{(m)}$$

uniformly. $\square$

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7 (Optional)

(a) For $\varphi \in \mathcal{D}(\mathbb{R})$ take $A > 0$ so $\operatorname{spt}(\varphi) \subset (-A, A)$ and note that for each $a \in (0, A)$,

$$\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right) \frac{\varphi(x)}{x}\, dx = \left(\int_{-A}^{-a} + \int_{a}^{A}\right) \frac{\varphi(x) - \varphi(0)}{x}\, dx$$

Here

$$\Phi(x) = \begin{cases} \dfrac{\varphi(x) - \varphi(0)}{x}, & 0 < |x| \leq A \\ \varphi'(0), & x = 0, \end{cases}$$

is continuous, so the improper integral defining $\langle \operatorname{pv}(\frac{1}{x}), \varphi \rangle$ is well-defined. It's then clear that $\operatorname{pv}(\frac{1}{x}) : \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ is linear, and since for $\varphi \in \mathcal{D}(-A, A)$ we have that

$$|\langle \operatorname{pv}(\tfrac{1}{x}), \varphi \rangle| \leq \max |\varphi'|$$

it follows that $\mathrm{pv}(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R})$ (of order at most 1). For $r>0$ we have

$$\langle d_r\, \mathrm{pv}(\tfrac{1}{x}), \varphi \rangle = \langle \mathrm{pv}(\tfrac{1}{x}), \tfrac{1}{r} d_{\frac{1}{r}} \varphi \rangle =$$

$$\lim_{a \to 0+} \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) \frac{\frac{1}{r}\varphi(\frac{x}{r})}{x}\, dx \underset{dy = \frac{dx}{r}}{\overset{y=\frac{x}{r}}{=}}$$

$$\lim_{a \to 0+} \left(\int_{-\infty}^{-\frac{a}{r}} + \int_{\frac{a}{r}}^{\infty} \right) \frac{\varphi(y)}{ry}\, dy = r^{-1} \langle \mathrm{pv}(\tfrac{1}{x}), \varphi \rangle$$

showing that $d_r\, \mathrm{pv}(\frac{1}{x}) = r^{-1}\, \mathrm{pv}(\frac{1}{x})$.

For $\varphi \in \mathcal{D}(\mathbb{R})$: $\langle \frac{d}{dx} \log|x|, \varphi \rangle = \langle \log|x|, -\varphi' \rangle =$

$$-\int_{-\infty}^{\infty} \log|x|\, \varphi'(x)\, dx = -\lim_{\varepsilon \to 0+} \left(\int_{-\frac{1}{\varepsilon}}^{-\varepsilon} + \int_{\varepsilon}^{\frac{1}{\varepsilon}} \right) \log|x|\, \varphi'(x)\, dx$$

$$\overset{\text{parts}}{=} -\lim_{\varepsilon \to 0+} \Big\{ \Big[\log|x| (\varphi(x) - \varphi(0)) \Big]_{x=-\frac{1}{\varepsilon}}^{x=-\varepsilon} - \int_{-\frac{1}{\varepsilon}}^{-\varepsilon} \frac{\varphi(x)-\varphi(0)}{x}\, dx$$

$$+ \Big[\log|x| (\varphi(x) - \varphi(0)) \Big]_{x=\varepsilon}^{x=\frac{1}{\varepsilon}} - \int_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{\varphi(x)-\varphi(x)}{x}\, dx \Big\} =$$

$$\lim_{\varepsilon \to 0+} \left(\int_{-\frac{1}{\varepsilon}}^{-\varepsilon} + \int_{\varepsilon}^{\frac{1}{\varepsilon}} \right) \frac{\varphi(x)-\varphi(0)}{x}\, dx =$$

$$\lim_{\varepsilon \to 0+} \left(\int_{-\frac{1}{\varepsilon}}^{-\varepsilon} + \int_{\varepsilon}^{\frac{1}{\varepsilon}} \right) \frac{\varphi(x)}{x}\, dx = \langle \mathrm{pv}(\tfrac{1}{x}), \varphi \rangle. \quad \square$$

<u>Note</u> $pr(\frac{1}{x})$ has order 1 : Assume not, then for some constant c we have

$$|\langle pr(\tfrac{1}{x}), \varphi\rangle| \leq c \max|\varphi| \quad \text{for } \varphi \in \mathcal{D}[-1,1].$$

Take $\varphi = \rho_\varepsilon * \mathbb{1}_{[\varepsilon, \frac{1}{2}]}$ for $0<\varepsilon<\frac{1}{2}$. Then

$$c = c\max|\varphi| \geqslant |\langle pr(\tfrac{1}{x}), \varphi\rangle| = \lim_{a\to 0+} \int_a^\infty \frac{\varphi(x)}{x}\,dx$$

$$= \int_0^\infty \frac{\varphi(x)}{x}\,dx > \int_{2\varepsilon}^{\frac{1}{2}-\varepsilon} \frac{dx}{x} = \log \frac{\frac{1}{2}-\varepsilon}{2\varepsilon} \quad ↯$$

<u>(b)</u> For $\varphi \in \mathcal{D}(\mathbb{R})$ we calculate:

$$\langle x\, pr(\tfrac{1}{x}), \varphi\rangle = \langle pr(\tfrac{1}{x}), x\varphi\rangle =$$

$$\lim_{a\to 0+} \left(\int_{-\infty}^{-a} + \int_a^\infty\right) \frac{x\varphi(x)}{x}\,dx = \int_{-\infty}^{\infty} \varphi(x)\,dx,$$

hence $x\, pr(\frac{1}{x}) = 1$.

Since the equation (2) is linear we get GS from (5): $u = pr(\frac{1}{x}) + c\delta_0$, $c \in \mathbb{R}$.
