Solutions to Sheet 3 (B4.3 MT18)

① Recall that for $u \in \mathcal{D}'(\Omega)$ its support $\mathrm{supp}(u)$ can be characterized as the smallest relatively closed subset A of Ω s.t. $u|_{\Omega \setminus A} = 0$. Take $\varphi \in \mathcal{D}(\Omega)$ with $\mathrm{supp}(\varphi) \subset \Omega \setminus \mathrm{supp}(u)$. Then for a multi-index $\alpha \in \mathbb{N}_0^n$ we have

$$\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle = 0$$

since clearly $D^\alpha \varphi|_{\Omega \setminus \mathrm{supp}(u)} \in \mathcal{D}(\Omega \setminus \mathrm{supp}(u))$.

Thus we must have too

$$\langle p(D)u, \varphi \rangle = \sum_{|\alpha| \leq k} c_\alpha \langle D^\alpha u, \varphi \rangle = 0$$

for all such φ, showing that $p(D)u|_{\Omega \setminus \mathrm{supp}(u)} = 0$ and therefore $\mathrm{supp}(p(D)u) \subseteq \mathrm{supp}(u)$.

→ BACKGROUND: Look up Peetre's characterization of PDO.

$\boxed{\text{EX}}$ Let $H : \mathbb{R} \to \mathbb{R}$ be the Heaviside function. Then $H \in \mathcal{D}'(\mathbb{R})$ and $\mathrm{supp}(H) = [0, \infty)$ but $H' = \delta_0$ has support $\{0\}$.

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② Write $x = (x_j, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and note that for $\xi_j \neq 0$ Fubini yields: $\hat{f}(\xi) = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} f(x) e^{-i x_j \xi_j} \, dx_j \, e^{-i x' \cdot \xi'} dx'$

$$\underset{x_j \mapsto x_j + \frac{\pi}{\xi_j}}{=} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} f(x + \tfrac{\pi}{\xi_j} e_j) e^{-i(x_j + \frac{\pi}{\xi_j})\xi_j} \, dx_j \, e^{-i x' \cdot \xi'} dx' =$$

$$e^{-i\pi} \int_{\mathbb{R}^n} f(x + \tfrac{\pi}{\xi_j} e_j) e^{-i x \cdot \xi} \, dx$$

$$= -\int_{\mathbb{R}^n} f\left(x + \frac{\pi}{\xi_j} e_j\right) e^{-ix\cdot\xi}\, dx.$$

Consequently, $\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n}\left(f(x) - f\left(x + \frac{\pi}{\xi_j} e_j\right)\right) e^{-ix\cdot\xi}\, dx$

and so

$$|\hat{f}(\xi)| \leq \frac{1}{2}\int_{\mathbb{R}^n} \left|f(x) - f\left(x + \frac{\pi}{\xi_j} e_j\right)\right| dx.$$

Let $\varepsilon > 0$. Because $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ we can find $g \in \mathcal{D}(\mathbb{R}^n)$ so $\|f - g\|_1 < \frac{\varepsilon}{2}$.

Then $|\hat{f}(\xi)| \leq |\hat{f}(\xi) - \hat{g}(\xi)| + |\hat{g}(\xi)| =$

$|(f-g)^\wedge(\xi)| + |\hat{g}(\xi)| \leq \|f-g\|_1 + |\hat{g}(\xi)| < \frac{\varepsilon}{2} + |\hat{g}(\xi)|$.

Take $R > 0$ so $\operatorname{spt}(g) \subset [-R, R]^n$. Since g is uniformly continuous we can find $\delta \in (0,1)$ so

$$|g(x) - g(y)| < \frac{\varepsilon}{(2(R+1))^n}$$

for $x, y \in \mathbb{R}^n$ with $|x - y| < \delta$.

Assume $\|\xi\|_\infty > \frac{\pi}{\delta}$. Then $\frac{\pi}{\|\xi\|_\infty} < \delta$, hence for $j \in \{1, \dots, n\}$ with $|\xi_j| = \|\xi\|_\infty$ we get by the above

$$|\hat{g}(\xi)| \leq \frac{1}{2}\int_{\mathbb{R}^n} \left|g(x) - g\left(x + \frac{\pi}{\xi_j} e_j\right)\right| dx$$

$$= \frac{1}{2}\int_{(-R-1, R+1)^n} \left|g(x) - g\left(x + \frac{\pi}{\xi_j} e_j\right)\right| dx$$

$$\leq \frac{1}{2}(2(R+1))^n \frac{\varepsilon}{(2(R+1))^n} = \frac{\varepsilon}{2}.$$

Consequently we get for all $\xi \in \mathbb{R}^n$ with $\|\xi\|_\infty > \frac{\pi}{\delta}$ that $|\hat{f}(\xi)| < \varepsilon$. □

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③ From Auxiliary Lemma for Fourier Inversion Formula in $\mathcal{S}$ we get

$$\hat{G}_t(\xi) = \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{4t}},$$

so by a Convolution Rule,

$$\widehat{G_s * G_t}(\xi) = \hat{G}_s(\xi)\,\hat{G}_t(\xi) = \left(\frac{\pi^2}{st}\right)^{\frac{n}{2}} e^{-\left(\frac{1}{4s}+\frac{1}{4t}\right)|\xi|^2}$$

Hence, applying the Auxilary Lemma once more,

$$\widehat{\widehat{G_s * G_t}}(\xi) = \mathcal{F}_{\xi\to x}\left\{ \left(\frac{\pi^2}{st}\right)^{\frac{n}{2}} e^{-\left(\frac{1}{4s}+\frac{1}{4t}\right)|\xi|^2} \right\}$$

$$= \left(\frac{\pi^2}{st}\right)^{\frac{n}{2}} \left(\frac{\pi}{\frac{1}{4s}+\frac{1}{4t}}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4\left(\frac{1}{4s}+\frac{1}{4t}\right)}}$$

$$= (2\pi)^n \left(\frac{\pi}{s+t}\right)^{\frac{n}{2}} e^{-\frac{st}{s+t}|x|^2}.$$

Finally from the Fourier Inversion Formula for $\mathcal{S}$, $\widehat{\widehat{(\cdot)}} = (2\pi)^n \widetilde{(\cdot)}$, so

$$G_s * G_t(x) = \left(\frac{\pi}{s+t}\right)^{\frac{n}{2}} e^{-\frac{st}{s+t}|x|^2}$$

<u>Note</u> If $H_t := (2\pi)^{-n}\hat{G}_t$, then we have the <u>semigroup property</u>: $H_s * H_t = H_{s+t}$, $\forall s,t > 0$.

(4) Write $-ax^2+bx+c = -a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}+c$ and use formula for Fourier transform of Gaussian (= Auxiliary Lemma):

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} e^{-a\left(x-\frac{b}{2a}\right)^2+\frac{b^2}{4a}+c-ix\xi}\,dx =$$

$x \to x - \frac{b}{2a}$

$$\int_{-\infty}^{\infty} e^{-a\left(x-\frac{b}{2a}\right)^2-ix\xi}\,dx \cdot e^{\frac{b^2}{4a}+c} =$$

$$\int_{-\infty}^{\infty} e^{-ax^2-i\left(x+\frac{b}{2a}\right)\xi}\,dx \; e^{\frac{b^2}{4a}+c} =$$

$$\int_{-\infty}^{\infty} e^{-ax^2-ix\xi}\,dx \; e^{-i\frac{b}{2a}\xi+\frac{b^2}{4a}+c} =$$

$$\left(\frac{\pi}{a}\right)^{\frac{1}{2}} e^{-\frac{|\xi|^2}{4a}-i\frac{b}{2a}\xi+\frac{b^2}{4a}+c}.$$

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(5) (a) BW – see LN Theorem 11.12.

(b) Clearly $\delta_x \in \mathcal{S}'(\mathbb{R}^n)$ and for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \hat{\delta}_x, \varphi\rangle = \langle \delta_x, \hat{\varphi}\rangle = \hat{\varphi}(x) =$$

$$\int_{\mathbb{R}^n} \varphi(y)\, e^{-ix\cdot y}\,dy = \langle e^{-ix\cdot(\cdot)}, \varphi\rangle,$$

so $\hat{\delta}_x = e^{-ix\cdot(\cdot)}$. From the Fourier

Inversion Formula in $\mathcal{S}'$,

$$(2\pi)^n \tilde{\delta}_x = \hat{\hat{\delta}}_x = \widehat{e^{-ix\cdot(\cdot)}}.$$

Now $e^{-ix\cdot\xi} = \lim_{j\to\infty} \mathbb{1}_{(-j,j)^n}(\xi)\, e^{-ix\cdot\xi}$

pointwise and boundedly in $\xi \in \mathbb{R}^n$, so by DCT also in $\mathcal{S}'(\mathbb{R}^n)$. Because $\mathcal{F}$ is $\mathcal{S}'$-continuous and $\mathbb{1}_{(-j,j)^n}(\cdot)\, e^{-ix\cdot(\cdot)} \in L^1(\mathbb{R}^n)$ we get

$$\mathcal{F}_{\xi\to y}\{e^{-ix\cdot\xi}\} = \lim_{j\to\infty} \int_{(-j,j)^n} e^{-ix\cdot\xi - iy\cdot\xi}\, d\xi.$$

Consequently,

$$\delta_x = (2\pi)^{-n} \tilde{\mathcal{F}}_{\xi\to y}\{e^{-ix\cdot\xi}\}$$

$$= \lim_{j\to\infty} (2\pi)^{-n} \int_{(-j,j)^n} e^{i(y-x)\cdot\xi}\, d\xi.$$

(c) If $\phi(x) = \sum_{|\alpha|\leq k} c_\alpha x^\alpha$, then we get from Lemma 11.3,

$$\overline{S}_{s,r}(\phi\varphi) \leq c\, \overline{S}_{s+k,r}(\varphi)$$

for all $s, r \in \mathbb{N}_0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where $c = c(q)$ is a constant that only depends on q. Put $q(x) := (1+|x|^2)^n p(x) \in \mathbb{C}[x]$.

Then for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$|p(x)\varphi(x)| \leqq c\,\overline{S}_{2n+\deg(p),0}(\varphi)\,(1+|x|^2)^{-n}$$

for all $x \in \mathbb{R}^n$. Using polar coordinates

$$\int_{\mathbb{R}^n} (1+|x|^2)^{-n}\,dx = \omega_{n-1}\int_0^\infty (1+r^2)^{-n} r^{n-1}\,dr < \infty$$

hence $p\varphi \in L^1(\mathbb{R}^n)$ and we may define

$$\langle p, \varphi \rangle := \int_{\mathbb{R}^n} p(x)\varphi(x)\,dx\,, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Hereby $\varphi \mapsto \langle p, \varphi \rangle$ is clearly linear and since

$$|\langle p, \varphi \rangle| \leqq c_1\,\overline{S}_{2n+\deg(p),0}(\varphi) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

with $c_1 = c\,\omega_{n-1}\int_0^\infty (1+r^2)^{-n} r^{n-1}\,dr$, $\quad p \in \mathcal{S}'(\mathbb{R}^n)$.

See also Lemma 12.3 (that you can also refer to for solving the problem).

Since $\delta_0 = (2\pi)^{-n} \tilde{\mathcal{F}}(\mathbb{1}_{\mathbb{R}^n})$ and $\tilde{\delta}_0 = \delta_0$,

$\mathcal{F}(\mathbb{1}_{\mathbb{R}^n}) = (2\pi)^n \delta_0$ (or by Fourier-Gelfand formula with $x=0$), hence by a Differentiation Rule,

$$\mathcal{F}(x_k) = iD_k((2\pi)^n \delta_0) = (2\pi)^n iD_k\delta_0 .$$

Consequently, $\mathcal{F}(p(x)) = (2\pi)^n p(iD)\delta_0$.

If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp}(\hat{u}) = \{0\}$, then by Theorem 13.13, $\hat{u} \in \operatorname{span}\{D^\alpha\delta_0 : \alpha \in \mathbb{N}_0^n\}$, That is, for some $k \in \mathbb{N}_0$ and $c_\alpha \in \mathbb{C}$, $|\alpha| \leq k$, $\hat{u} = \sum_{|\alpha| \leq k} c_\alpha D^\alpha \delta_0 = \sum_{|\alpha| \leq k} (-i)^{|\alpha|} c_\alpha (iD)^\alpha \delta_0$

$= (2\pi)^n p(iD)\delta_0$ provided $p(x) = \sum_{|\alpha| \leq k} \frac{(-i)^{|\alpha|} c_\alpha}{(2\pi)^n} x^\alpha$.

The result above yields $\hat{p} = \hat{u}$, and so by the Fourier Inversion Formula in $\mathcal{S}'$ we get $u = p$.

(d) If $u \in \mathcal{S}'(\mathbb{R}^n)$, then we say that u is real-valued if $\langle u, \varphi \rangle \in \mathbb{R}$ when $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is real-valued.

For $u \in \mathcal{S}'(\mathbb{R}^n)$ we define $\bar{u} \in \mathcal{S}'(\mathbb{R}^n)$ by the adjoint identity scheme: for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \bar{u}, \varphi \rangle := \overline{\langle u, \bar{\varphi} \rangle}.$$

The adjoint identity being: for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi \bar{\psi} \, dx = \overline{\int_{\mathbb{R}^n} \bar{\varphi} \psi \, dx}.$$

We have for $f \in L^1(\mathbb{R}^n)$ that

$$\hat{f}(-\xi) - \overline{\hat{f}(\xi)} = \int_{\mathbb{R}^n} \left(f(x) - \overline{f(x)} \right) e^{ix\cdot\xi} \, dx.$$

Consequently, if f is real-valued, then $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$ for all ξ. Conversely, if

$$0 = \int_{\mathbb{R}^n} \left(f(x) - \overline{f(x)} \right) e^{ix\cdot\xi} \, dx \quad \forall \xi,$$

then by the Fourier Inversion Formula in $\mathcal{S}'$ $f(x) - \overline{f(x)} = 0$ for a.e. $x \in \mathbb{R}^n$, that is, f is real-valued.

General distributions:

Claim. $u \in \mathcal{S}'(\mathbb{R}^n)$ real-valued iff $\tilde{\hat{u}} = \overline{\hat{u}}$.

$\boxed{\text{Pf.}}$ First note that u is real-valued iff $\bar{u} = u$: for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ real-valued,

$$\langle \bar{u} - u, \varphi \rangle = -2i \operatorname{Im} \langle u, \varphi \rangle,$$

and for a complex-valued distribution $v \in \mathcal{S}'(\mathbb{R}^n)$ we have $v = 0$ iff $\langle v, \varphi \rangle = 0$ $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$ that are real-valued. Next, we calculate for real-valued $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \tilde{\hat{u}} - \bar{\hat{u}}, \varphi \rangle = \overline{\langle \bar{u} - u, \hat{\varphi} \rangle}. \square$$

(e) From the Fourier Inversion Formula $\mathcal{F}^{-1} = (2\pi)^{-n} \tilde{\mathcal{F}}$ both in $\mathcal{S}$ and in $\mathcal{S}'$. Thus $\mathcal{F}^2 = (2\pi)^n \widetilde{(\cdot)}$, and so $\mathcal{F}^4 = (2\pi)^{2n} I$ both in $\mathcal{S}$ and in $\mathcal{S}'$. Now if $\lambda \in \mathbb{C}$ is an eigenvalue for $\mathcal{F}$ on $\mathcal{S}$ (or on $\mathcal{S}'$), then we find $\varphi \in \mathcal{S} \setminus \{0\}$ (or $\varphi \in \mathcal{S}' \setminus \{0\}$) with $\mathcal{F}\varphi = \lambda \varphi$. But then $\lambda^4 \varphi = \mathcal{F}^4 \varphi = (2\pi)^{2n} \varphi$, and since $\varphi \neq 0$, $\lambda^4 = (2\pi)^{2n}$ follows, and therefore

$$\lambda \in \left\{ \pm (2\pi)^{\frac{n}{2}}, \pm (2\pi)^{\frac{n}{2}} i \right\}.$$

6 Recall that $\langle d_r \hat{u}, \varphi \rangle = \langle \hat{u}, \frac{1}{r^n} d_{\frac{1}{r}} \varphi \rangle$
$= \frac{1}{r^n} \langle u, \widehat{d_{\frac{1}{r}} \varphi} \rangle$ and calculating we find
$\widehat{d_{\frac{1}{r}} \varphi}(\xi) = r^n \hat{\varphi}(r\xi)$, hence $\langle d_r \hat{u}, \varphi \rangle = \langle u, d_r \hat{\varphi} \rangle$.

By assumption $d_r u = r^\alpha u$ so

$$\langle d_r u, \hat{\varphi} \rangle = \langle u, \tfrac{1}{r^n} d_{\frac{1}{r}} \hat{\varphi} \rangle = r^\alpha \langle u, \hat{\varphi} \rangle,$$

that is, substituting r by $\frac{1}{r}$,

$$\langle u, d_r \hat{\varphi} \rangle = r^{-n-\alpha} \langle u, \hat{\varphi} \rangle = r^{-n-\alpha} \langle \hat{u}, \varphi \rangle.$$

Thus

$$\langle d_r \hat{u}, \varphi \rangle = r^{-n-\alpha} \langle \hat{u}, \varphi \rangle,$$

as required.

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7 (Optional)

Since $|x|^{\alpha+2} = |x|^{\alpha+2} \mathbb{1}_{\{|x|<1\}} + |x|^{\alpha+2} \mathbb{1}_{\{|x| \geq 1\}}$

$\in (L^1 + L^\infty)(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ as $-n-1 < \alpha < -2$

also $D_1 |x|^{\alpha+2} = (\alpha+2) x_1 |x|^\alpha \in \mathcal{S}'(\mathbb{R}^n)$.

(This can be justified by use of Theorem 2.4.1 in Strichartz' book.)

From Lemma 13.9 we have

$$\mathcal{F}_{x\to\xi}\left(|x|^{\alpha+2}\right) = c\,|\xi|^{-n-2-\alpha}$$

for some constant $c = c(n,\alpha) \in \mathbb{R}$. By a Differentiation Rule:

$$\mathcal{F}_{x\to\xi}\left(x_1|x|^{\alpha}\right) = \frac{1}{\alpha+2}\,(-i\xi_1)\,\mathcal{F}_{x\to\xi}\left(|x|^{\alpha+2}\right)$$

$$= -\frac{ci}{\alpha+2}\,\xi_1\,|\xi|^{-n-2-\alpha}.$$

⑧ (Optional)

For $\varphi \in \mathcal{S}(\mathbb{R})$ we have for all $a>0$ that $\frac{\varphi(x)}{x} \in L^1(\mathbb{R}\setminus(-a,a))$. In particular,

$$\int_{\mathbb{R}\setminus(-1,1)} \left|\frac{\varphi(x)}{x}\right| dx \leqq 2\int_1^\infty \frac{dx}{x^2} S_{0,0}(\varphi) = 2S_{0,0}(\varphi).$$

For $0<a<1$ we have since $\frac{\varphi(0)}{x}$ is odd that

$$\left|\left(\int_{-1}^{-a} + \int_a^1\right)\frac{\varphi(x)}{x}dx\right| = \left|\left(\int_{-1}^{-a} + \int_a^1\right)\frac{\varphi(x)-\varphi(0)}{x}dx\right|$$

$\leqq 2S_{0,1}(\varphi)$, and since the limit exists, we have shown that

$$\left|\left\langle \mathrm{pv}\left(\frac{1}{x}\right), \varphi\right\rangle\right| \leqq 2\bar{S}_{1,1}(\varphi) \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}).$$

Thus $\mathrm{pv}\left(\frac{1}{x}\right) \in \mathcal{S}'(\mathbb{R})$.

Note that

$$f_j(\xi) = \left(\int_{-j}^{-\frac{1}{j}} + \int_{\frac{1}{j}}^{j}\right)\frac{e^{-ix\cdot\xi}}{x}dx \xrightarrow[j]{} \widehat{\mathrm{pv}\left(\frac{1}{x}\right)}(\xi)$$

in $\mathcal{S}'(\mathbb{R})$. But $f_j'(\xi) = -i\left(\int_{-j}^{-\frac{1}{j}} + \int_{\frac{1}{j}}^{j}\right)e^{-ix\xi}dx$

$$\xrightarrow[j]{} -i\mathcal{F}(1) = -i\cdot 2\pi\delta_0 = -2\pi i\,\delta_0$$

But then $\widehat{\mathrm{pv}\left(\frac{1}{x}\right)}(\xi) = -2\pi i\, H + c$,

where H is the Heaviside function and $c \in \mathbb{C}$ is a constant. Because $\mathrm{pv}\left(\frac{1}{x}\right)$ is (-1)-homogeneous and odd, $\widehat{\mathrm{pv}\left(\frac{1}{x}\right)}$ must be 0-homogeneous and odd. But then $c = \pi i$ and hence

$$\widehat{\mathrm{pv}\left(\frac{1}{x}\right)}(\xi) = -\pi i \,\mathrm{sgn}(\xi).$$