Solutions to sheet 4 (B4.3 MT18/HT19)

(1) For $\varphi \in \mathcal{S}(\mathbb{R})$ and $\mathbb{1}_{(-t,t)} \in L^1(\mathbb{R})$ the Product Rule gives $\int_{\mathbb{R}} \hat{\varphi}\, \mathbb{1}_{(-t,t)}\, dx = \int_{\mathbb{R}} \varphi\, \widehat{\mathbb{1}_{(-t,t)}}\, dx$,

and so

$$(*) \quad \int_{-t}^{t} \hat{\varphi}\, dx = 2\int_{-\infty}^{\infty} \varphi(x) \frac{\sin(tx)}{x}\, dx$$

holds for all $t>0$, $\varphi \in \mathcal{S}(\mathbb{R})$.

Since $\mathbb{1}_{(-t,t)} \xrightarrow[t\to\infty]{} \mathbb{1}_{\mathbb{R}}$ in $\mathcal{S}'(\mathbb{R})$ and $\mathcal{F}$ is $\mathcal{S}'$-continuous,

$$\langle \widehat{\mathbb{1}_{\mathbb{R}}}, \varphi \rangle = \lim_{t\to\infty} 2\int_{-\infty}^{\infty} \varphi(x) \frac{\sin(tx)}{x}\, dx$$

for each $\varphi \in \mathcal{S}(\mathbb{R})$. From $\hat{\delta}_0 = \mathbb{1}_{\mathbb{R}}$ and the Fourier Inversion Formula in $\mathcal{S}'$,

$2\pi \tilde{\delta}_0 = \hat{\hat{\delta}}_0 = \widehat{\mathbb{1}_{\mathbb{R}}}$, hence, as $\tilde{\delta}_0 = \delta_0$,

$$\langle 2\pi \delta_0, \varphi \rangle = \lim_{t\to\infty} 2\int_{-\infty}^{\infty} \varphi(x) \frac{\sin(tx)}{x}\, dx$$

and the conclusion $\frac{\sin(tx)}{x} \xrightarrow[t\to\infty]{} \pi \delta_0$ in $\mathcal{S}'(\mathbb{R})$ follows.

② Clearly $|x|^k f \in L^1(\mathbb{R})$ for all $k \in \mathbb{N}_0$ so $\hat{f} \in C^k(\mathbb{R})$ with $\hat{f}^{(k)}(\xi) = \int_{-\infty}^{\infty} f(x)(-ix)^k e^{-ix\xi}\,dx$,

hence $\left|\hat{f}^{(k)}(\xi)\right| \leq \int_{-\infty}^{\infty} |x|^k |f(x)|\,dx \leq 2\int_0^{\infty} x^k e^{-x}\,dx$

$= 2 \cdot k!$.

If $\operatorname{spt}(\hat{f}) \neq \mathbb{R}$ then we may select $\xi_0 \in \partial(\operatorname{spt}(\hat{f}))$. Clearly $\hat{f}^{(k)}(\xi_0) = 0$ for all k, so by Taylor's formula

$$\hat{f}(\xi) = \sum_{k=0}^{n} \frac{\hat{f}^{(k)}(\xi_0)}{k!}(\xi - \xi_0)^k + R_n(\xi, \xi_0)$$

$$= R_n(\xi, \xi_0)$$

where $R_n(\xi, \xi_0) = \dfrac{\hat{f}^{(n+1)}(\xi_0 + \theta(\xi - \xi_0))}{(n+1)!}(\xi - \xi_0)^{n+1}$

for some $\theta = \theta_n(\xi, \xi_0) \in (0,1)$. Now we may estimate

$$|R_n(\xi, \xi_0)| \leq \frac{2 \cdot (n+1)!}{(n+1)!}|\xi - \xi_0|^{n+1} = 2|\xi - \xi_0|^{n+1}$$

and so if $|\xi - \xi_0| < 1$, then

$R_n(\xi, \xi_0) \to 0$ as $n \to \infty$.

Consequently, $\hat{f}(\xi) = 0$ for all $\xi \in (\xi_0 - 1, \xi_0 + 1)$.

But then $\xi_0 \notin \partial(\operatorname{spt}(\hat{f}))$ and we conclude that either $\operatorname{spt}(\hat{f}) = \emptyset$ or $\operatorname{spt}(\hat{f}) = \mathbb{R}$.

Distribution Theory and Fourier Analysis. Sheet 4 solutions.

3 Clearly $f \in L^1(\mathbb{R}^n)$ (any $n \in \mathbb{N}$) so

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix.\xi} dx = \int_{\mathbb{R}^n} e^{-|x| - ix.\xi} dx.$$

(a) $\underline{n=1}$: $\hat{f}(\xi) = \int_{-\infty}^{0} e^{(1-i\xi)x} dx + \int_{0}^{\infty} e^{-(1+i\xi)x} dx \overset{FTC}{=}$

$\frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+|\xi|^2}$, and so by Fourier Inversion Formula in $\mathcal{S}'(\mathbb{R})$ (note $f \notin \mathcal{S}$)

$$e^{-|x|} = \frac{1}{2\pi} \tilde{\mathcal{F}}(\hat{f})(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+|\xi|^2} e^{ix\xi} d\xi.$$

Take $\lambda = |x|$ yields required identity.

(b) Now for $\lambda \geq 0$:

$$e^{-\lambda} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+|\xi|^2} e^{i\lambda\xi} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-(1+|\xi|^2)t} dt \, e^{i\lambda\xi} d\xi$$

$$\overset{\text{Fubini}}{=} \frac{1}{\pi} \int_{0}^{\infty} \underbrace{\int_{-\infty}^{\infty} e^{-t|\xi|^2} e^{i\lambda\xi} d\xi}_{\mathcal{F}(e^{-t|\xi|^2})(-\lambda)} e^{-t} dt$$

$G_t = e^{-t|\cdot|^2}$

see proof for Fourier Inversion in $\mathcal{S}$

$$= \frac{1}{\pi} \int_{0}^{\infty} \left(\frac{\pi}{t}\right)^{\frac{1}{2}} e^{-\frac{\lambda^2}{4t}} e^{-t} dt$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{\pi t}} e^{-t-\frac{\lambda^2}{4t}} dt, \text{ as required.}$$

(c) $$\hat{f}(\xi) \overset{(b)}{=} \int_{\mathbb{R}^n} \int_{0}^{\infty} \frac{1}{\sqrt{\pi t}} e^{-t-\frac{|x|^2}{4t}} dt \, e^{-ix.\xi} dx$$

$$\overset{\text{Fubini}}{=} \int_{0}^{\infty} \frac{1}{\sqrt{\pi t}} e^{-t} \hat{G}_{(4t)^{-1}}(\xi) dt =$$

$$\int_0^\infty \frac{1}{\sqrt{\pi t}}\, e^{-t} \left(\frac{\pi}{\frac{1}{4t}}\right)^{\frac{n}{2}} e^{-\frac{4t|\xi|^2}{4}}\, dt =$$

$$2^n \pi^{\frac{n-1}{2}} \int_0^\infty t^{\frac{n-1}{2}} e^{-\cancel{4}t|\xi|^2-t}\, dt \overset{s=(1+|\xi|^2)t}{=}$$

$$2^n \pi^{\frac{n-1}{2}} \int_0^\infty \left(\frac{s}{1+|\xi|^2}\right)^{\frac{n-1}{2}} e^{-s} \frac{ds}{1+|\xi|^2} =$$

$$2^n \pi^{\frac{n-1}{2}} \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}} \int_0^\infty s^{\frac{n-1}{2}} e^{-s}\, ds =$$

$$2^n \pi^{\frac{n-1}{2}}\, \Gamma\!\left(\frac{n+1}{2}\right) \frac{1}{(1+|\xi|^2)^{\frac{n+1}{2}}}.$$

---

(4)

(a) $g \in \mathcal{S}'(\mathbb{R}^n)$ by Lemma 12.2. Calculating in polar coordinates we see that

$$\int_{\mathbb{R}^n} |g(x)|\, dx = \omega_{n-1} \int_0^\infty (1+r^2)^{-\frac{\alpha}{2}} r^{n-1}\, dr < \infty$$

iff $-2\frac{\alpha}{2} + n - 1 < -1$, so iff $\alpha > n$.

(b) $$\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-(1+|x|^2)t}\, dt \overset{s=(1+|x|^2)t}{=}$$

$$\int_0^\infty \left(\frac{s}{1+|x|^2}\right)^{\frac{\alpha}{2}-1} e^{-s} \frac{ds}{1+|x|^2} = (1+|x|^2)^{-\frac{\alpha}{2}} \int_0^\infty s^{\frac{\alpha}{2}-1} e^{-s}\, ds$$

$$= \Gamma\!\left(\frac{\alpha}{2}\right) g(x), \quad \text{so} \quad c = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \in (0,\infty).$$

(c) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then using (b) and Fubini we calculate:

$$\langle \hat{g}, \varphi \rangle = \langle g, \hat{\varphi} \rangle = c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \int_{\mathbb{R}^n} e^{-t|x|^2} \hat{\varphi}(x)\,dx\,dt$$

Recall $G_t(x) = e^{-t|x|^2}$, $\hat{G}_t(x) = \left(\frac{\pi}{t}\right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ (see Theorem 10.3) so by the Product Rule:

$$\int_{\mathbb{R}^n} G_t \hat{\varphi}\,dx = \int_{\mathbb{R}^n} \hat{G}_t \varphi\,dx,$$

hence

$$\langle \hat{g}, \varphi \rangle = c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \int_{\mathbb{R}^n} \hat{G}_t(x)\varphi(x)\,dx\,dt$$

$$= \int_{\mathbb{R}^n} \left(c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \hat{G}_t(x)\,dt \right) \varphi(x)\,dx$$

by Fubini, since

$$c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \hat{G}_t(x)\,dt > 0 \quad \forall x$$

and

$$\int_{\mathbb{R}^n} c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \hat{G}_t(x)\,dt\,dx \overset{\text{Tonelli}}{=}$$

$$c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \int_{\mathbb{R}^n} \hat{G}_t(x)\,dx\,dt \underset{\substack{\uparrow \\ \text{See pf} \\ \text{for Th. 10.3}}}{=} c\int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} (2\pi)^n\,dt$$

$$= c\,\Gamma\!\left(\tfrac{\alpha}{2}\right)(2\pi)^n = (2\pi)^n < \infty.$$

Thus $\hat{g}(x) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} e^{-t} \hat{G}_t(x)\,dt$,

a positive integrable function, as required.

(5)

(a) For $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$(1+x^2)|\varphi(x)| \leqq 2\bar{S}_{2,0}(\varphi) \quad \forall x \in \mathbb{R},$$

and so for $\varepsilon \geqslant 0$,

$$|\operatorname{Log}(x \pm i\varepsilon)\varphi(x)| \leqq 2\bar{S}_{2,0}(\varphi)\,\frac{|\log(x^2+\varepsilon^2)^{\frac{1}{2}}| + \pi}{1+x^2}$$

$$\leqq 2\bar{S}_{2,0}(\varphi)\,\frac{|\log|x|| + \log(|x|+1) + \pi}{1+x^2}$$

for $x \in \mathbb{R}\setminus\{0\}$, $\varepsilon \geqq 0$. Note that

$$2\bar{S}_{2,0}(\varphi)\,\frac{|\log|x|| + \log(|x|+1) + \pi}{1+x^2} \in L^1(\mathbb{R})$$

and

$$\operatorname{Log}(x \pm i\varepsilon)\varphi(x) \xrightarrow[\varepsilon \to 0+]{} \operatorname{Log}(x \pm i0)\varphi(x)$$

pointwise in $x \in \mathbb{R}\setminus\{0\}$, where

$$\operatorname{Log}(x \pm i0) := \log|x| + i\begin{cases} 0 & \text{if } x>0 \\ \pm\pi & \text{if } x<0. \end{cases}$$

Thus $\operatorname{Log}(x \pm i0) \in \mathcal{S}'(\mathbb{R})$ and the ε-uniform bound yields for $\varphi \in \mathcal{S}(\mathbb{R})$:

$$|\langle \operatorname{Log}(x \pm i0), \varphi\rangle| \leqq c\,\bar{S}_{2,0}(\varphi),$$

where $$c = 2\int_{-\infty}^{\infty} \frac{|\log|x|| + \log(|x|+1) + \pi}{1+x^2}\,dx.$$

(b) For $\varphi \in \mathcal{S}(\mathbb{R})$ the definition gives

$$\langle (x \pm i0)^{-k}, \varphi \rangle = -\frac{1}{(k-1)!} \langle \operatorname{Log}(x \pm i0), \varphi^{(k)} \rangle$$

and so

$$|\langle (x \pm i0)^{-k}, \varphi \rangle| \leq \frac{c}{(k-1)!} \bar{S}_{2,0}(\varphi^{(k)})$$

$$\leq \frac{c}{(k-1)!} \bar{S}_{2,k}(\varphi).$$

If also $\varphi^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k$ we find by partial integration:

$$\langle (x \pm i0)^{-k}, \varphi \rangle = -\frac{1}{(k-1)!} \int_{-\infty}^{\infty} \operatorname{Log}(x \pm i0) \varphi^{(k)}(x) \, dx$$

$$= -\frac{1}{(k-1)!} \left\{ \left(\int_{-\infty}^{0} + \int_{0}^{\infty} \right) \log|x| \, \varphi^{(k)}(x) \pm i\pi \int_{-\infty}^{0} \varphi^{(k)}(x) dx \right\}$$

$$= +\frac{1}{(k-1)!} \left(\int_{-\infty}^{0} + \int_{0}^{\infty} \right) \frac{1}{x} \varphi^{(k-1)}(x) \, dx \quad + 0$$

$$= \int_{-\infty}^{\infty} \frac{1}{x^k} \varphi(x) \, dx,$$ as required.

(c) Inspection gives that

$$\operatorname{Log}(x + i0) - \operatorname{Log}(x - i0) = 2\pi i H(-x)$$

for all $x \in \mathbb{R} \setminus \{0\}$, and thus

$$\operatorname{Log}(x + i0) - \operatorname{Log}(x - i0) = 2\pi i \tilde{H} \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Differentiating with respect to x k times and multiplying by $\frac{(-1)^{k-1}}{(k-1)!}$ we find

$$\frac{(-1)^{k-1}}{(k-1)!} \frac{d^k}{dx^k}\left\{ \mathrm{Log}(x+i0) - \mathrm{Log}(x-i0)\right\} =$$

$$2\pi i \frac{(-1)^{k-1}}{(k-1)!} \frac{d^k}{dx^k} \tilde{H} = 2\pi i \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}}(-\delta_0) =$$

$$2\pi i \frac{(-1)^k}{(k-1)!} \delta_0^{(k-1)} \text{, as required.}$$

(d) For $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\langle x(x \pm i0)^{-1}, \varphi \rangle = \langle \tfrac{d}{dx} \mathrm{Log}(x \pm i0), x\varphi \rangle$$

$$= -\int_{-\infty}^{\infty} \mathrm{Log}(x \pm i0)(x\varphi)' \, dx =$$

$$-\int_{-\infty}^{\infty} \log|x| \, (x\varphi)' \, dx - i\int_{-\infty}^{0} (\pm\pi)(x\varphi)' \, dx$$

$$\overset{\text{parts on } (-\infty,0) \text{ and } (0,\infty)}{=} \int_{-\infty}^{\infty} \frac{1}{x} x\varphi \, dx - 0 = \int_{-\infty}^{\infty} \varphi \, dx,$$

so $x(x \pm i0)^{-1} = 1$ in $\mathcal{S}'(\mathbb{R})$.

Deduction is clear (and so one must be careful with the associative rule).

From $x(x \pm i0)^{-1} = 1$ in $\mathcal{S}'(\mathbb{R})$ we get by applying $x\frac{d}{dx}$ that

$$0 = x\frac{d}{dx}\left(x(x \pm i0)^{-1}\right) \overset{\text{Leibniz}}{=} x\left((x \pm i0)^{-1} + \right.$$

$$\left. + x\left(-(x \pm i0)^{-2}\right)\right) = x(x \pm i0)^{-1} - x^2(x \pm i0)^{-2},$$

so $x^2(x \pm i0)^{-2} = 1$ in $\mathcal{S}'(\mathbb{R})$.

Now suppose that $x^k(x \pm i0)^{-k} = 1$ in $\mathcal{S}'(\mathbb{R})$ for some $k \in \mathbb{N}$. Then we find

$$0 = x\frac{d}{dx}\left(x^k (x \pm i0)^{-k}\right) \overset{\text{Leibniz}}{=}$$

$$x\left(kx^{k-1}(x \pm i0)^{-k} + x^k\left(-k(x \pm i0)^{-k-1}\right)\right)$$

$$= kx^k(x \pm i0)^{-k} - kx^{k+1}(x \pm i0)^{-k-1},$$

and induction concludes the proof.

(e) $$\mathcal{F}_{x\to\xi}(H_\varepsilon(x)) = \int_0^\infty e^{-\varepsilon x - ix\xi}\,dx = \frac{1}{\varepsilon + i\xi} =$$

$$-i(\xi - i\varepsilon)^{-1} = -i\frac{d}{d\xi}\operatorname{Log}(\xi - i\varepsilon), \quad \varepsilon > 0.$$

Since $H_\varepsilon \longrightarrow H$ and $\operatorname{Log}(\xi - i\varepsilon) \longrightarrow \operatorname{Log}(\xi - i0)$ as $\varepsilon \to 0+$ in $\mathcal{S}'(\mathbb{R})$ and $\mathcal{F}, \frac{d}{d\xi}$ are $\mathcal{S}'$-continuous, $\mathcal{F}_{x\to\xi}(H(x)) = -i(\xi - i0)^{-1}$.

We also have

$$\mathcal{F}_{x\to\xi}(\tilde{H}) = \mathcal{F}_{x\to-\xi}(H) = -i(-\xi-i0)^{-1} =$$

$i(\xi+i0)^{-1}$, so by Fourier Inversion Formula in $\mathcal{S}'$,

$$\mathcal{F}_{x\to\xi}\left((x+i0)^{-1}\right) = \mathcal{F}\left(-i\,\mathcal{F}(\tilde{H})\right)$$

$$= -2\pi i\, H(\xi).$$

19/15

$$\cancel{\leq \frac{1}{2}(2(R+1))^n \frac{\varepsilon}{(2(R+1))^n} = \frac{\varepsilon}{2}}$$

~~Consequently we get for all~~ $\cancel{\xi \in \mathbb{R}^n}$ ~~with~~ $\cancel{\|\xi\|_\infty > \frac{\pi}{\delta}}$ ~~that~~ $\cancel{|\hat{f}(\xi)| \leq \varepsilon}$.

---

6 (a) Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then for all $m, n \in \mathbb{N}_0$ we have that

$$c_{m,n} := \sup_x (1+|x|)^n |\varphi^{(m)}(x)| < \infty,$$

and so $|\varphi^{(m)}(x)| \leq \dfrac{c_{m,n}}{(1+|x|)^n}$, $\forall x \in \mathbb{R}$.

For each $k \in \mathbb{Z}$, $x \in \mathbb{R}$:

$$|\varphi^{(m)}(x - 2\pi k)| \leq \frac{c_{m,n}}{(1+|2\pi|k|-|x||)^n}.$$

Take $n=2$ and fix $r>0$. Then for $k \in \mathbb{Z}$, $|x| \leq r$ we have

$$|\varphi^{(m)}(x-2\pi k)| \leq \frac{c_{m,2}}{(1+|2\pi|k|-r|)^2}$$

provided $|k| \geq \frac{r}{2\pi}$. Since clearly $\sum_{k\in\mathbb{Z}} \frac{c_{m,2}}{(1+|2\pi|k|-r|)^2} < \infty$ we deduce from the Weierstraß M-test that $\sum_{k \in \mathbb{Z}} \varphi^{(m)}(x-2\pi k)$ is uniformly convergent in $x \in [-r,r]$. Therefore $P\varphi$ is well-defined and C^∞. It is clear that $P\varphi$ is 2π-periodic.

From Prelims we know that a smooth 2π-periodic function is equal to its Fourier series:

$$P\varphi(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}, \quad x \in \mathbb{R},$$

where

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} P\varphi(x)\, e^{-ikx}\,dx \overset{\text{uniform cvg}}{=}$$

$$y = x - 2\pi n$$

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} \varphi(x-2\pi n)\, e^{-ikx}\,dx =$$

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \int_{-(2n+1)\pi}^{-(2n-1)\pi} \varphi(y)\, e^{-iky - ik\cdot 2\pi n}\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(y)e^{-iky}\,dy$$

$$= \frac{1}{2\pi}\hat{\varphi}(k).$$

(b) From (a) we get for $x \in \mathbb{R}$:

$$\sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\hat{\varphi}(k)\, e^{ikx} = P\varphi(x) = \sum_{k\in\mathbb{Z}} \varphi(x-2\pi k)$$

and so in particular for $x=0$. that

$$\sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\hat{\varphi}(k) = \sum_{k\in\mathbb{Z}} \varphi(2\pi k).$$

(c) Take $\varphi = G_t = e^{-t|\cdot|^2}$ and recall that $\hat{G}_t(\xi) = \left(\frac{\pi}{t}\right)^{\frac{1}{2}} e^{-\frac{|\xi|^2}{4t}}$, so that (4) yields the required identity.

7 (Optional)

By the Identity theorem for holomorphic functions we deduce that since $F \not\equiv 0$ $Z_F = \{z_j : j \in J\}$ has no cluster points in $\mathbb{C}$, and the corresponding multiplicities $\{m_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$. Consequently the set $\{j \in J : |z_j| < k\}$ must be finite for each $k \in \mathbb{N}$ by Bolzano-Weierstraß. We can there= fore take $r_j > 0$ for each $j \in J$ so $\{B_{r_j}(z_j) : j \in J\}$ are mutually disjoint.

In $B_{r_j}(z_j)$ we can by virtue of Taylor's theorem write $F(z) = (z - z_j)^{m_j} F_j(z)$, where $F_j : B_{r_j}(z_j) \to \mathbb{C}$ is holomorphic and nonzero.

Now $\log|F_j| = \frac{1}{2}\log(F_j \overline{F_j})$ and $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ so as $\log|F_j| \in C^\infty(B_{r_j}(z_j))$ we calculate

$$\Delta \log|F_j| = 2\frac{\partial}{\partial z}\left(\frac{1}{F_j \overline{F_j}}\left(F_j \frac{\partial}{\partial \bar{z}}\overline{F_j} + \left(\frac{\partial}{\partial \bar{z}} F_j\right)\overline{F_j}\right)\right) \overset{\text{Cauchy Riemann}}{=}$$

$$2\frac{\partial}{\partial z}\left(\frac{1}{\overline{F_j}}\overline{F_j'}\right) = 2\frac{\partial}{\partial z}\left\{\overline{\left(\frac{F_j'}{F_j}\right)}\right\} = \overline{2\frac{\partial}{\partial \bar{z}}\left(\frac{F_j'}{F_j}\right)} = 0$$

Thus $\log|F_j|$ is harmonic on $B_{r_j}(z_j)$.

Since on $B_{r_j}(z_j)$, $F(z) = (z-z_j)^{m} F_j(z)$, we have clearly $\log|F| \in L^1(B_{r_j}(z_j)) \subset \mathcal{S}'(B_{r_j}(z_j))$ hence

$$\Delta \log|F| = \Delta\left(m_j \log|z-z_j| + \log|F_j(z)| \right) =$$

$2\pi m_j \delta_{z_j}$, since $\frac{1}{2\pi}\log|z|$ is fundamental solution for Δ on $\mathbb{C}$.

Now $\log|F| \in L^1_{loc}(\mathbb{C}) \subset \mathcal{D}'(\mathbb{C})$ and if $\varphi \in \mathcal{D}(\mathbb{C})$ and $K = \{ j \in J : \mathrm{spt}(\varphi) \cap B_{r_j}(z_j) \neq \emptyset \}$ (so K is finite) we put for $z \in \mathrm{spt}(\varphi) \setminus \bigcup_{j\in K} B_{r_j}(z_j)$,

$$r(z) = \frac{1}{2} \min_{j \in J} |z - z_j| .$$

Then $\{ B_{r(z)}(z) : z \in \mathrm{spt}(\varphi) \setminus \bigcup_{j \in K} B_{r_j}(z_j) \} \cup \{ B_{r_j}(z_j) \}_{j \in K}$

is an open cover of the set $\mathrm{spt}(\varphi)$. By compactness it admits a finite subcover, say

$$B_{r(z_s)}(z_s), \quad 1 \leq s \leq m, \quad B_{r_j}(z_j), \quad j \in K.$$

Put $B^*_s = B_{r(z_s)}(z_s)$, $B_j = B_{r_j}(z_j)$, and use Theorem 2.5 to find $\phi^*_s, \phi_j \in \mathcal{D}\left(\bigcup_{s=1}^{m} B^*_s \cup \bigcup_{j\in K} B_j\right)$ with $\mathrm{spt}(\phi^*_s) \subset B^*_s$, $0 \leq \phi^*_s \leq 1$, $\mathrm{spt}(\phi_j) \subset B_j$, $0 \leq \phi_j \leq 1$

and so $\sum_{s=1}^{m} \phi_s^* + \sum_{j\in K} \phi_j = 1$ on $\mathrm{spt}(\varphi)$.

Now $\varphi = \sum_{s=1}^{m} \phi_s^* \varphi + \sum_{j\in K} \phi_j \varphi$ and $\langle \Delta \log|F|, \varphi\rangle =$
$\sum_{s=1}^{m} \langle \Delta \log|F|, \phi_s^* \varphi\rangle + \sum_{j\in K} \langle \Delta \log|F|, \phi_j \varphi\rangle$.

Since F is holomorphic and nonzero on B_s^* we have $\langle \Delta \log|F|, \phi_s^* \varphi\rangle = 0$.

On $B_{r_j}(z_j)$ we have $\Delta \log|F| = 2\pi m_j \delta_{z_j}$, hence

$$\langle \Delta \log|F|, \varphi\rangle = \sum_{j\in K} \langle 2\pi m_j \delta_{z_j}, \phi_j \varphi\rangle.$$

Because $\mathrm{spt}(\phi_j) \subset B_j$ and the $\{B_j\}_{j\in J}$ are mutually disjoint we get

$$\sum_{j\in K} \langle 2\pi m_j \delta_{z_j}, \phi_j \varphi\rangle = \sum_{k\in K} \langle \sum_{j\in J} 2\pi m_j \delta_{z_j}, \phi_k \varphi\rangle$$

$$= \langle \sum_{j\in J} 2\pi m_j \delta_{z_j}, \Big(\sum_{k\in K} \phi_k\Big)\varphi\rangle.$$

Since $\Big(\sum_{s=1}^{m} \phi_s^* \varphi\Big)(z_j) = 0$ for all $j\in J$ we have shown that $\Delta \log|F| = \sum_{j\in J} 2\pi m_j \delta_{z_j}$. $\square$