B4.1 Functional Analysis I MT 2018: Problem Sheet 2

When not specified, the scalar field $\mathbb{F}$ may be assumed to be either $\mathbb{R}$ or $\mathbb{C}$.

1. Let $(X, \|\cdot\|)$ be a normed space

(i) Let $S, T \in L(X)$ be invertible. Prove that also ST is invertible.

(ii) Let $S \in L(X)$ be algebraically invertible, i.e. so that there exists a map $T : X \to X$ so that $TS = ST = \mathrm{Id}$ which we call the algebraic inverse and denote by S^{-1}. Prove that $S^{-1} \in L(X)$ if and only if

$$(\star) \qquad \exists \delta > 0 \text{ so that } \forall x \in X \text{ we have } \|S(x)\| \geqslant \delta \|x\|.$$

Conversely show that $(\star)$ is violated if and only if there exist $x_n \in X$ with $\|x_n\| = 1$ so that $Sx_n \to 0$.

2. Let $X = C([-1, 1])$ equipped with the sup-norm and define $f : X \to \mathbb{R}$ by

$$f(\phi) = \int_0^1 \phi(t)dt - \int_{-1}^0 \phi(t)dt.$$

Prove that

(i) $f \in X^* = L(X, \mathbb{F})$

(ii) $\|f\| = 2$

(iii) There exists no $\phi \in X$ so that $\|\phi\|_{sup} = 1$ and $f(\phi) = 2$.

3. Let X be the subspace of ℓ^2 that is defined by

$$X := \{x \in \ell^2 : (jx_j) \in \ell^1\} \subset \ell^2$$

and define

$$P(x_1, x_2, \ldots) = (\sum_{j=1}^{\infty} jx_j, 0, 0, \ldots).$$

(i) Check that P is a linear map from X to X such that $P^2 = P$.

(ii) Is X (equipped with the ℓ^2-norm) a Banach space?

(iii) Is P bounded?

Carefully justify your answers to (ii) and (iii).

4. Let $X = C[a, b]$ equipped with the sup norm. For x in X define Tx by

$$(Tx)(t) = \int_a^t x(s)\,\mathrm{d}s \quad (t \in [a, b]).$$

(i) Prove that this defines a bounded linear operator $T \in L(X)$ and that $\|T\| = b - a$.

(ii) Find a function $k : \{\, (s, t) \mid a \leqslant s \leqslant t \leqslant b \,\} \to \mathbb{R}$ so that

$$(T^2x)(t) = \int_a^t k(s, t)x(s)\,\mathrm{d}s.$$

and determine $\|T^2\|$.

(iii) *(Optional)* Determine $k_n : \{\, (s, t) \mid a \leqslant s \leqslant t \leqslant b \,\} \to \mathbb{R}$ so that

$$(T^n x)(t) = \int_a^t k_n(s, t)x(s)\,\mathrm{d}s.$$

5. Let $t_{jk} \in \mathbb{R}$ $(j, k = 1, 2, \ldots)$ and denote by $e^{(k)}$, $k = 1, 2, \ldots$ the sequences $e^{(k)} = (\delta_{kj})_{j\in\mathbb{N}}$. Show that if $\sup_{k,j} |t_{jk}| < \infty$ then there exists a bounded linear operator $T : \ell^1 \to \ell^\infty$ so that

$$(\star\star) \qquad Te^{(k)} = \sum_{j=1}^{\infty} t_{jk} e^{(j)}.$$

Conversely, show that if there is a bounded linear operator $T : \ell^1 \to \ell^\infty$ so that $(\star\star)$ holds, then we must have that $\sup_{k,j} |t_{jk}| < \infty$ and indeed $\sup_{k,j} |t_{jk}| = \|T\|$.

6. (i) Let $\alpha = (\alpha_j)$ be a fixed bounded sequence. Define M_α by

$$M_\alpha \colon (x_1, x_2, x_3, \ldots) \mapsto (\alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots).$$

Prove that $M_\alpha \in L(\ell^\infty)$. For which values of α is M_α injective? Prove that M_α has a bounded inverse if and only if $\inf_j |\alpha_j| > 0$.

(ii) Let X be the space of real polynomials on $[0, 1]$ regarded as a subspace of the Banach space $C[0, 1]$ of continuous functions equipped with the sup norm. For $k = 0, 1, 2, \ldots$, let m_k be the k-th monomial, i.e.

$$m_k(t) = t^k \quad (t \in [0, 1]).$$

Define $T \colon X \to X$ by letting

$$Tm_k = \frac{1}{k+1} m_k$$

and extend T to X by linearity.

(a) Give an integral expression for Tx for a general $x \in X$ and use this expession to prove that T can be extended to a bounded linear operator $\tilde{T} \in L(C[0, 1])$ with $\|\tilde{T}\| = 1$.

(b) Prove that $T : X \to X$ is a bijection but that its inverse is not bounded.

(c) Is $\tilde{T} : C[0, 1] \to C[0, 1]$ still injective? Is it still surjective?