Topology & Groups

Michaelmas 2016

Question Sheet 1

1. For each of the following groups G and generating sets S, draw the resulting Cayley graph:

   (i) $G = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, $S = \{(1,0),(0,1)\}$;

   (ii) $G =$ the dihedral group of order 8, viewed as the symmetries of the square, and $S = \{\sigma, \tau\}$, where σ is a rotation of order 4, and τ is a reflection through an axis joining opposite sides;

   (iii) $G =$ the free group on two generators a and b, and $S = \{a, b\}$. [You will need to be familiar with free groups from Part A Group Theory for this question. If you did not do that course, then skip this part of the question.]

2. Let Γ be the Cayley graph of a group G with respect to a generating set S.

   (i) Show that the following is a metric on G: $d(g_1, g_2) =$ the shortest number of edges in a path in Γ joining the vertex labelled g_1 to the vertex labelled g_2.

   (ii) Show that $d(g_1, g_2)$ equals the smallest non-negative integer n such that

   $$g_2 = g_1 s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_n^{\epsilon_n}$$

   where each $s_i \in S$ and $\epsilon_i \in \{-1, 1\}$.

   (iii) Let $\mathrm{Isom}(G)$ be the group of isometries of G with this metric. Prove that G can be realised as a subgroup of $\mathrm{Isom}(G)$.

   (iv) Find an example where $G \subsetneqq \mathrm{Isom}(G)$.

3. Recall that the surface S_g with g handles can be constructed from a $4g$-sided polygon by identifying its sides in pairs. Show that S_g can be given the structure of a cell complex, with a single 0-cell, $2g$ 1-cells and a single 2-cell.

4. Give a cell structure for the 3-torus $S^1 \times S^1 \times S^1$. Try to use as few cells as possible. [It is possible to use a single 0-cell, three 1-cells, three 2-cells and one 3-cell.]