Topology & Groups

Michaelmas 2016

Question Sheet 3

Questions with an asterisk $*$ beside them are optional.

You may assume throughout this sheet that $\pi_1(S^1) \cong \mathbb{Z}$, and that a generator for $\pi_1(S^1)$ is represented by the loop $t \mapsto e^{2\pi i t}$.

1. Show that for a space X, the following three conditions are equivalent:

   (i) Every map $S^1 \to X$ is homotopic to a constant map.

   (ii) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

   (iii) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

   Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, 'homotopic' means 'homotopic without regard to basepoints'.]

2. Let X and Y be spaces with basepoints x_0 and y_0. Show that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

   Deduce that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.

3. A *retraction* of a space X onto a subspace A is a map $r\colon X \to A$ such that $ri = \mathrm{id}_A$, where $i\colon A \to X$ is the inclusion map.

(i) Prove that there is no retraction map $r\colon D^2 \to S^1$.

(ii) Our aim here is to show that any map $f\colon D^2 \to D^2$ has a fixed point. Suppose that, on the contrary, f has no fixed point; in other words $f(x) \neq x$ for all $x \in D^2$. Use the pairs $(x, f(x))$ to construct a retraction $D^2 \to S^1$. Thus, we deduce that any map $D^2 \to D^2$ must have a fixed point.

   This fact has many applications outside of topology. For example, it can be used to show that certain differential equations always have a solution.

4. For $n > 2$, prove that no two of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^n$ are homeomorphic.

$*$ 5. Show that there is no retraction of a Möbius band onto its boundary.